# Лекция 1.2 — "1.2 Слабая, инициальная, финальная, фактор-топология, топология порожденная семейством отображений и топология произведения"

## Введение в нелинейный функциональный анализ

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So far we have dealt with a set that had an a priori given topology.
In this lection we consider two situations where a set is given a topology which is natural under the circumstances.
These are the "weak topology" and the "quotient topology" .
Let us briefly describe the starting point for the introduction of these two topologies.
So let $X$ be our set. For the weak topology the situation is the following.

**Слабая или инициальная топология, или топология порожденная семейством отображений**

We are given a family $\{Y_i, f_i\}_{i \in I}$ of pairs, each consisting of a topological space $Y_i$ and a map $f_i : X \to Y_i$.
Any topology of $X$ that makes all the $f_i$ continuous, is said to be admissible.
Evidently, the set of admissible topologies on $X$ is nonempty, since the discrete topology is such a topology.
We will see that there exists a topology $\omega$ on $X$ such that every admissible topology is stronger or equal to $\omega$.

## Фактор-топология

For the quotient topology, the setting is reversed.
We are given a family of pairs

$$\{Y_i, f_i\}_{i \in I}$$

where each $Y_i$ is a topological space and $f_i : Y_i \to X$.
As before we call a topology on $X$ admissible if it makes all the $f_i$ continuous.
This time the indiscrete (trivial topology), certainly is admissible.
We will see that there exists a topology $\tau$ on $X$ such that the admissible topologies are those topologies which are weaker or equal to $\tau$.

Before moving to the detailed examination of the weak and quotient topologies, let us remark that the intersection of any nonempty family of topologies on a set $X$ is a topology on $X$ (it is the greatest lower bound for the partial order determined by the relation "weaker than").
However, the union of two topologies need not be a topology. First let us examine the weak topology.

DEFINITION 1.2.1 Let $\{Y_i, f_i\}_{i\in I}$ ($I$ is an arbitrary index set) be a family of pairs where $Y_i$ is a topological space and $f_i : X \to Y_i$ is a map. The "weak topology"(or "initial topology") on $X$, generated by the family $\{f_i\}_{i\in I}$, is the weakest topology on $X$ that makes all the functions $\{f_i\}_{i\in I}$ continuous. So it is the topology generated by

$$\mathcal{F} = \{f_i^{-1}(V) \ : \ i \in I, \ V \subseteq Y_i \text{ is open}\} \ \ (i.e.\ \mathcal{F} = \bigcup_{i\in I} f^{-1}(\tau_{Y_i})).$$

This is a subbase for the weak topology. In fact we can economize in the definition of the subbase for the weak topology and take

$$\mathcal{F}_1 = \{f_i^{-1}(V) \ : \ i \in I, \ V \subseteq Y_i \text{ is subbasic open}\}.$$

This too is a subbase for the weak topology. We denote the weak topology on $X$ generated by the family $\{f_i\}_{i\in I}$ by $\omega(X, \{f_i\}_{i\in I})$ or simply by $\omega$ if no confusion is possible. Of course a base for the weak topology is given by all the sets of the form $\bigcap_{i=1}^n f_i^{-1}(V_i)$ with $V_i \in \tau_{Y_i}$ and $n \geq 1$ an arbitrary integer.

PROPOSITION 1.2.2 If a set $X$ is furnished with the weak topology $\omega(X, \{f_i\}_{i \in I})$, then $x_\alpha \to x$ if and only if for all $i \in I$ we have $f(x_\alpha) \to f(x)$ in $Y_i$.

Proof.

$\Rightarrow$: Because each $f_i$ is $\omega$-continuous, $x_\alpha \xrightarrow{\omega} x$ implies that $f(x_\alpha) \xrightarrow{\omega} f(x)$ for every $i \in I$.
$\Leftarrow$: Let $U = \cap_{i=1}^n f_i^{-1}(V_i)$ be a basic neighborhood of $x$ (Definition 1.2.1). Since by hypothesis for each $i \in I$, $f(x_\alpha) \to f(x)$ in $Y_i$, we can find $\alpha_i$ such that for each $\alpha \geq \alpha_i$, we have $x_\alpha \in f^{-1}(V_i)$. Choose $\widehat{\alpha} \geq \alpha_i$ for all $i \in \{1, \ldots, n\}$ (Definition 1.1.18). Then for $\alpha \geq \widehat{\alpha}$, we have $x_\alpha \in U$ and so $x_\alpha \xrightarrow{\omega} x$.

PROPOSITION 1.2.3 If $Z$ is a topological space, $X$ is a set furnished with the weak topology $\omega(X, \{f_i\}_{i\in I})$ and $g : Z \to X$, then $g$ is continuous if and only if $f_i \circ g$ is continuous for each $i \in I$.

Proof.

$\Rightarrow$: Immediate from Corollary 1.1.30(b).
$\Leftarrow$: If $z_\alpha \to z$ in $Z$, then $f_i(g(z_\alpha)) \to f_i(g(z))$ for each $i \in I$. By virtue of Proposition 1.2.2, this implies that $g(z_\alpha) \to g(z)$ in $X$, hence $g$ is continuous (Theorem 1.1.29).

EXAMPLES 1.2.4 (a) Let $Y$ be a topological space, $X \subseteq Y$ and let $i\colon X \to Y$ be the map $i(x) = x$ (embedding of $X$ into $Y$) . The trivial family $(Y, i)$ induces a weak topology on $X$. A subbase for this topology is given by

$$\mathcal{F} = \{i^{-1}(V) \ : \ V \subseteq Y \text{ is open}\} = \{V \cap X \ : \ V \subseteq Y \text{ is open}\}.$$

In fact $\mathcal{F}$ is already a topology, the subbase (or relative) topology on $X$ (Example 1.1.3(e)).

(b)

(b) Let $T$ be a set and let $X$ be any set of functions $f : T \to \mathbb{R}$. For every $t \in T$, let $e_t\colon X \to \mathbb{R}$ be defined by $e_t(f) = f(t)$, $f \in X$. Then the family $(\mathbb{R}, \{e_t\}_{t\in T})$ induces a weak topology on $X$. By Proposition 1.2.2, $f_\alpha \xrightarrow{\omega} f$ if and only if $f_\alpha(t) \to f(t)$ for all $t \in T$ (i.e. weak convergence is equivalent to pointwise convergence).

REMARK 1.2.5 When the spaces $Y_i = \mathbb{R},\ i \in I$ , then a subbase of $\omega(X, \{f_i\}_{i \in I})$ is given by all sets of the form

$$U_i(x, \varepsilon) \ = \ \{y \in X \ : \ f_i(y) - f_i(x)| < \varepsilon\},$$

where $i \in I$, $x \in X$ and $\varepsilon > 0$.

Now we will investigate a little the separation character of the weak topology. For this purpose we need the following definition.

DEFINITION 1.2.6 Let $X$ be a set and $\{f_i\}_{i\in I}$ be a family of functions each of which has domain $X$. We say that the family $\{f_i\}_{i\in I}$ is "separating"(or "total" ), if for each pair of points $x, y \in X$, $x \neq y$ there exist $i \in I$ such that $f_i(x) \neq f_i(y)$.

PROPOSITION 1.2.7 If $X$ is a set, $f_i\colon X \to Y_i$, $i \in I$, a separating family of functions and for each $i \in I$, $Y_i$ is Hausdorff then $X$ furnished with the weak topology $\omega(X, \{f_i\}_{i\in I})$ (denoted by $X_\omega$ is also Hausdorff).

Proof.

Let $x_1, x_2 \in X$ $x_1 \neq x_2$. Since by hypothesis the family is separating, we can find an $i \in I$ such that $f(x_1) \neq f(x_2)$. Because $Y_i$ is Hausdorff, there exist $V_{\in}\mathcal{N}(f_i(x_k))$, $k = 1, 2$ such that $V_1 \cap V_2 = \emptyset$. Then $f_i^{-1}(V_k)$, $k = 1, 2$, are disjoint weak neighborhoods of $x_1, x_2$. So $X_\omega$ is Hausdorff. □

Let $X$ be a set and $A \subseteq X$. Let $f_i \colon X \to \mathbb{R}$, $\in I$, be a family of functions. On $A$ we have two topologies. One is the relative weak topology generated by $\{f_i\}_{i\in I}$ (i.e. the restriction of the weak topology $\omega(X, \{f_i\}_{i\in I})$ on A) and the other is the weak topology generated by $f_i|_A, I \in I$.
It is natural to ask whether these two topologies are the same.
The next proposition shows that the answer to this question is affirmative.

PROPOSITION 1.2.8 If $X$ is a set, $f_i \colon X \to \mathbb{R}$, $i \in I$, are functions and $A \subseteq X$,

then

$$\omega(X, \{f_i\})|_A = \omega(A, \{f_i|_A\}_{i\in I}).$$

Proof.

Using Proposition 1.2.2 we can check that the two topologies have the same a convergent nets and so are identical.□

We will prove some more simple results about the weak topology. For this we need the following definition-notation.

DEFINITION 1.2.9 Let $(X, \tau)$ be a topological space. We introduce the following two sets:

(a) $C(X, \tau)$ or $C(X)$ is the space of all continuous functions $f: X \to \mathbb{R}$;

(b) $C_b(X, \tau)$ or $C_b(X)$ is the space of all bounded continuous functions $f: X \to \mathbb{R}$.

PROPOSITION 1.2.10 $\omega(X, C(X)) = \omega(X, C_b(X))$.

Proof.

Evidently $\omega(X, C_b(X)) \subseteq \omega(X, C(X))$. Let $U$ be a subbasic set for $\omega(X, C(X))$. Then

$$U = U(f, x, \varepsilon) = \{y \in X \ : \ |f(y) - f(x)| < \varepsilon|\},$$

where $f \in C(X)$, $x \in X$ and $\varepsilon > 0$. Set

$$g(z) = \min\{f(x) + \varepsilon, \ \max\{f(x) - \varepsilon, f(x)\}\}.$$

Clearly $g \in C_b(X)$ and $U(g, x, \varepsilon) = U(f, x, \varepsilon)$. Therefore $\omega(X, C(X)) \subseteq \omega(X, C_b(X))$ and so finally equality follows. $\square$

In the next theorem we use weak topologies to characterize completely regular spaces.

THEOREM 1.2.11 A topological space $(X, \tau)$ is completely regular if and only if $\tau = \omega(X, C(X)) = \omega(X, C_b(X))$.

Proof.

$\Rightarrow$: First note that $\omega(X, C(X)) \subseteq \tau$. Let $x \in U \in \tau$ and since $X$ is completely regular, we can find $f \in C(X)$ such that $f(x) = 0$ and $f|_{U^c} = 1$. Then the set $V = \{y \in X \ : \ f(y) < 0\}$ is a $\omega(X, C(X))$-neighborhood of $x$ and $V \subseteq U$. Therefore $U$ is $\omega(X, C(X))$-open and so $\tau \subseteq \omega(X, C(X))$, hence $\tau = \omega(X, C(X)) = \omega(X, C_b(X))$. $\Leftarrow$: Let $C \subseteq X$ closed and $x \notin C$. Since $C^c$ is weakly open, we can find $U \subseteq C^c$ with $U = \bigcap_{k=1}^{n} \{y \in X \ : |f_k(y) - f_k(x)| < 1\}$, where $f \in C(X)$. Let $g_k(x) = \min\{1, \ |f_k(z) - f_k(x)|\}, \quad k \in \{1, \dots, n\}$ and set $g = \max_{1 \le k \le n} g_k$. Then $g \colon X \to [0, 1]$ is continuous and satisfies $g(x) = 0$ and $g|_C = 1$. Therefore $X$ is completely regular (Definition 1.1.55). □

COROLLARY 1.2.12 If $(X, \tau)$ is a completely regular space,

then

$$x_\alpha \longrightarrow x$$

in $X$ if and only if

$$f(x_\alpha) \longrightarrow f(x)$$

for all $f \in C_b(X)$.

Now we will present a very useful weak topology, which is the "product topology" .
In fact we will show that up to a homeomorphism this is the only Hausdorff weak topology.
So let $I$ be any index set and let $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces. Let

$$X = \prod_{i \in I} X_i$$

the Cartesian product $X_i$. A generic point $x \in X$ is described by

$$x = (x_i)_{i \in I}$$

with $x_i \in X_i$ for all $i \in I$.
If $X_i = V$ for all $i \in I$, then we write $\prod_{i \in I} X_i = V^I$ and is the set of all functions from $I$ to $V$.
Let $p_i$ be the projection from $X$ onto the $i^{th}$-coordinate space $X_i$, i.e.$p_i(x) = p_i((x_j)_{j \in I}) = x_i$ for any $\in I$.
Then letting $f_i = p_i$, we can define the weak topology $\omega(X, \{f_i\}_{i \in I})$.

DEFINITION 1.2.13 Let $(X_i, \tau_i)_{i \in I}$ be topological spaces and

$$X = \prod_{i \in I} X_i.$$

The "product topology"on $X$ is the weak topology $\omega(X, \{p_i\}_{i \in I})$, i.e. is the weakest topology making all the coordinate projections continuous.

EXAMPLE 1.2.14 Take the Cartesian product of just two spaces, each a copy of the real line with the usual topology. Then the product topology equals the usual topology on $\mathbb{R}^2$ (the metric topology known from multivariable calculus), since either the set of all open balls or the set of all open rectangles gives a base for the same a topology. (An open ball is the union of all the open rectangles it includes and conversely).

REMARK 1.2.15 From Definition 1.2.13 and Definition 1.2.1 it follows that a base for the product topology of $\prod_{i\in I} X_i$ is the collection of sets $\prod_{i\in I} U_i$, where all the sets $U_i$ are nonempty open subsets of $X_I$ with $U_i = X_i$ for all but a finite number of indices of $i \in I$. Hence a product of open sets need not be open. However, a product of closed sets is always closed. Indeed note that $\prod_{i\in I} C_i = \bigcap_{i\in I} p_i^{-1}(C_i)$ and the latter set is closed being the intersection of closed sets. Also the coordinate projection maps $p_i : X = \prod_{j\in I} X_j \to X_i,\ i \in I$, are continuous open maps. From Proposition 1.2.7, we see that if the coordinate (factor) spaces $X$ are Hausdorff then so is $X = \prod_{i\in I} X_i$ with the product topology. In fact $X = \prod_{i\in I} X_i$ with the product topology is Hausdorff if and only if each $X_i$ is Hausdorff. Similarly, $X = \prod_{i\in I} X_i$ with the product topology is regular (resp. completely regular) if and only if each factor space $X_i$, $i \in I$, is regular (resp. completely regular) . However, the product of normal spaces need not be normal. On the other hand, if $X = \prod_{i\in I} X_i$ with the product topology is normal, then each factor space is normal. Finally, it is easy to verify that for any sets $A_i \subseteq X_i$, $i \in I$, we have $\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i}$ (from this follows immediately that the product of closed sets is closed) .

As we already mentioned any Hausdorff weak topology is homeomorphic to a product topology. Let us make this more precise.

PROPOSITION 1.2.16 If $X$ is a set and $f_i\colon X \to Y_i$, $i \in I$, a separating family of functions into the Hausdorff topological spaces $Y_i$, $i \in I$,

then

the map $x \to (f_i(x))_{i\in I}$ is a homeomorphism from $X$ with the $\omega(X, \{f_i\}_{i\in I})$ topology onto a subspace of $\prod_{i\in I} Y_i$ with the product topology.

**Proof.**

Since by hypothesis $\{f_i\}_{i\in I}$ is separating, the map $x \mapsto (f_i(x))_{i\in I}$ is one-to-one. Also by virtue of Proposition 1.2.2, it follows that $x_\alpha \to x$ in $X$ if and only if $(f_i(x_\alpha))_{i\in I} \to (f_i(x))_{i\in I}$ in $\prod_{i\in I} Y_i$. So both $f$ and $f^{-1}$ are continuous.

Using this proposition 1.2.16 together with Remark 1.2.15 which says that regularity and complete regularity are preserved by products and since both separation properties are hereditary (see Remarks 1.1.54 and 1.1.57), we obtain the following strengthened version of Proposition 1.2.7.

PROPOSITION 1.2.17 If $X$ is a set and $f_i\colon X \to Y_i$, $i \in I$, is a separating family of functions with range spaces $Y_i$, $i \in I$, which are regular (resp. completely regular), then the weak topology $\omega(X, \{f_i\}_{i\in I})$ is also regular (resp. completely regular).

Recall that given a function $f\colon X \to Y$ its graph is the set $Gr\, f = \{(x,y) \in X \times Y \;:\; y = f(x)\}$.

PROPOSITION 1.2.18 If $X$, $Y$ are topological spaces, $Y$ is Hausdorff and $f\colon X \to Y$ is continuous, then $Gr\, f$ is closed in $X \times Y$ with the product topology.

**Proof.**

Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ be a net of elements in $Gr\, f$ and assume that $(x_\alpha, y_\alpha) \to (x,y)$. Then since $f$ is continuous $f(x_\alpha) \to f(x)$. Also $y_\alpha \to y$ and for all $\alpha \in D$,$y_\alpha = f(x_\alpha)$. In the Hausdorff space the limits are unique. Hence $y = f(x)$, i.e. $(x,y) \in Gr\, f$ , which proves the closedness of $Gr\, f$.

For $\mathbb{R}^*$-valued semicontinuous functions the following sets are important.

DEFINITION 1.2.19 Given a set $X$ and a function $f : X \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$, the epigraph (resp. hypograph) of $f$ is the set defined by

$$epi\, f = \{(x, \lambda) \in X \times \mathbb{R} \ : \ f(x) \leq \lambda\}$$

(resp.

$$hyp\, f = \{(x, \lambda) \in X \times \mathbb{R} \ : \ f(x) \geq \lambda\}$$

.

PROPOSITION 1.2.20 If $X$ is a topological space and $f: X \to \mathbb{R}^*$,

then

$f$ is lower semicontinuous if and only if *epi* $f$ is closed in $X \times \mathbb{R}$ with the product topology.

Similarly for upper semicontinuous functions with *epi* $f$ replaced by *hyp* $f$.

$\Rightarrow$: Let $\{(x_\alpha, \lambda_\alpha)\}_{\alpha \in D}$ be a net in *epi* $f$ and assume that $(x_\alpha, \lambda_\alpha) \to (x, \lambda)$ in $X \times \mathbb{R}$. We have $f(x_\alpha) \leq \lambda_\alpha$ and because $f$ is lower semicontinuous, $f(x) \leq \liminf f(x_\alpha) \leq \lambda$ (Proposition 1.1.37) and so $(x, \lambda) \in$ *epi* $f$, which proves that *epi* $f \subseteq X \times \mathbb{R}$ is closed.
$\Leftarrow$: Consider the function $\varphi: X \times \mathbb{R} \to \mathbb{R}^*$ defined by $\varphi(x, \lambda) = f(x) - \lambda$. Then for every $\mu \in \mathbb{R}$,

$$\{(x, \lambda) \in X \times \mathbb{R} \ : \ \varphi(x, \lambda) \leq \mu\} = epi\ f + (0, \mu),$$

hence the set is closed and this by .Definition 1.1.34 implies that $\varphi$ is lower

Directly from the definition of the product topology as a weak topology (Definition 1.2.13) and Proposition 1.2.3, we have

PROPOSITION 1.2.21 If $X$, $(\{Y_i\}_{i\in I}$ are topological spaces, $f_i\colon X \to Y_i$, $i \in I$, are maps and $f\colon X \to \prod_{i\in I} Y_i$, is defined by $f(x) = (f_i(x))_{i\in I}$, then $f$ is continuous if and only if each $f_i$ is continuous.

# Определение Фактор-топологии

So let $X$ be a set, $\{Y_i\}_{i\in I}$ a family of topological spaces and $f_i\colon Y_i \to X$, $i \in I$, a family of maps. We are looking for the strongest topology on $X$ which makes all the $f_i$ continuous.

DEFINITION 1.2.22 The "quotient topology" on $X$, denoted by $\tau_q$ is the topology defined by

$$\tau_q = \{U \subseteq X \ : \ \text{for every } i \in I, \ \ f_i^{-1}(U) \ \text{ is open in } \ Y_i\}.$$

This is the strongest (largest) topology on $X$ making all the $f_i$ continuous.

Свойства Фактор-топологии для одного отображения $f\colon Y \to X$. Удобно ввести обозначение

$$A^* = f^{-1}(f(A)),$$

где $A$ — произвольное множество в $Y$. Выполняются соотношения

$$A^* \supseteq A, \quad f(A^*) = f(A), \quad A^{**} = A$$

Отметим пять свойств Фактор-топологии:

(1) Для каждого множества $A \subseteq Y$ имеет место равенство

$$f(Y \backslash A^*) = f(Y) \backslash f(A)$$

(2) Множество $f(A)$, где $A \subseteq Y$, открыто тогда и только тогда, когда открыто множество $A^*$.

# Продолжение свойств финальной топологии

(3) Отображение $f$ открыто тогда и только тогда, когда вместе с каждым открытым множеством $U$ в $X$ открыто и множество $U^*$. Если $f$ — открытое отображение и $U$ пробегает базу окрестностей точки $x$, то система множеств $f(U)$ образует базу окрестностей точки $f(x)$.

(4) Если $f$ открыто и отображает $Y$ на $X$, то множество $f(A)$ замкнуто тогда и только тогда, когда замкнуто множество $A^*$.

(5) Отображение $g\colon X \to Z$ непрерывно тогда и только тогда, когда отображение $g \circ f\colon Y \to Z$ непрерывно.

REMARK 1.2.23 It is easy to check that $\tau_q$ given in Definition 1.2.22 is indeed a topology. If $I$ ia singleton, $Y_i = Y$ and $f_i = f\colon Y \to X$ is surjective, then $f$ is called the "quotient map", if $X$ is endowed with the $\tau_q$-topology.

Convergence in $\tau_q$ is not as easily described as convergence in the weak topology.
On the other hand continuity has a nice description analogous to Proposition 1.2.3.

PROPOSITION 1.2.24 If $X$ ,$\{Y_i\}_{i\in I}$, $\{f_i\}_{i\in I}$ and $\tau_q$ are above, $Z$ is a topological space and $g\colon X \to Z$, then $g$ is $\tau_q$-continuous if and only if for every $i \in I$, $g \circ f_i\colon Y_i \to Z$ are continuous.

Proof.

$\Rightarrow$: Obvious from Corollary 1.1.30(b).
$\Leftarrow$: Let $V$ be an open set in $Z$ and let $U = g^{-1}(V)$. For every $i \in I$,

$$f_i^{-1}(U) = (g \circ f_i)^{-1}(U)$$

and so $f_i^{-1}(U)$ is open in $Y$. Then $U \in \tau_q$ and so $g$ is $\tau_q$-continuous.

Now we restrict ourselves to the case when $I$ is singleton, $Y_i = Y$ and $f_i = f\colon Y \to X$ is surjective, which is what we have usually in practice.

PROPOSITION 1.2.25 If $Y$, $X$ are topological spaces and $f\colon Y \to X$ is continuous, surjective open map, then $f$ is a quotient map (Remark 1.2.23).

Proof.

Let $\tau$ be the topology on $X$. Then $\tau \subseteq \tau_q$. Let $U \in \tau_q$. Then $f^{-1}(U)$ is open in $Y$ (from Definition 1.2.22), hence

$$U = f(f^{-1}(U)) \in \tau,$$

since $f$ is open. So $\tau = \tau_q$.

REMARK 1.2.26 If

$$Y = \prod_{i \in I} Y_i,$$

then each $Y_i$ has the quotient topology by the projection map $p_i\colon Y \to Y_i$.

PROPOSITION 1.2.27 If $X$, $Y$ are topological spaces and $f\colon X \to Y$ is a continuous, surjective map which sends closed sets into closed sets, then $f$ is a quotient map.

Proof.

Let $\tau$ be the topology of $Y$. We have $\tau \subseteq \tau_q$. Let $U \in \tau_q$. Then $f^{-1}(U)$ is open, hence $f^{-1}(U)^c$ is closed. Therefore $f(f^{-1}(U)^c) = C$ is $\tau$-closed. But $C = U^c$, hence $\tau = \tau_q$.

Intuitively, a quotient map is the nearest thing possible to a homeomorphism.

EXAMPLE 1.2.28 Let $Y = [0, 1]$ and $X = S_1 = \partial B_1$ (being the boundary of the unit ball in $\mathbb{R}^2$). It is well known that $Y$ and $X$ are not homeomorphic. Define $f: Y \to X$ by $f(t) = e^{2\pi it}$ (identify $\mathbb{R}^2$ with the complex p1ane). Then $f$ is a homeomorphism on $(0, 1)$ and is continuous on $[0, 1]$. It is a quotient map (this will become clear in the next section, Proposition 1.3.8) . However, it is not an open map (consider the open set $[0, 1/2)$).

This example describes a sense in which the unit sphere $X$ is constructed out of the line segment $Y = [0, 1]$. If we ignore the topology of $X$, take the given map $f$ from $Y$ onto $X$ and equip $X$ with the quotient topology, we obtain the unit sphere. Since $f$ is one-to-one except of $f(0) = f(1)$, there is a sense in which $X$ is constructed out of $Y$ by identifying the end-points and disturbing the topology as little as possible. Now we formalize this process.

Let $f\colon Y \to X$ be surjective and define a relation $\backsim$ on $Y$ by setting $y \backsim y'$ if and only if $f(y) = f(y')$. This is an equivalence relation which partitions $Y$ into a collection of disjoint subsets, namely the subsets $q(y) = \{y' \in Y \ : \ y \backsim y'\}$. Clearly $z \in q(y)$ if and only if $q(z) = q(y)$. The collection of all subsets $\{q(y) \ : \ y \in Y\}$ is called the quotient space of $Y$ by $\backsim$ and it is denoted by $Y/\backsim$. The map $q\colon Y \to Y/\backsim$ is called the "quotient map". Each point of $Y/\backsim$ is a subset of $Y$. Also $f$ is constant on each set $q(y) = \{y' \in Y \ : \ f(y) \backsim f(y')\}$ and so we may define the map $g\colon Y/\backsim \to X$ by $g(q(y)) = f(y)$ (since $q(y) = q(y')$ implies $f(y) = f(y')$). So we have the following commutative diagram in Figure 1.1

The function $g$ is onto (since $f$ is) and, which is more important, is one-to-one. Indeed, if $g(u) = g(v)$, where $u, v \in Y/\backsim$, we have $u = q(y), v = q(y')$. Then $f(y) = g(u) = g(v) = f(y')$, hence $y \backsim y'$ and so $q(y) = q(y')$, that is $u = v$. We topologize $Y/\backsim$. Namely we give it the quotient topology by the quotient map $q\colon Y \to Y/\backsim$. This is called the "quotient topology generated by the relation" $\backsim$.

In the next theorem, we summarize all this discussion and we show that every quotient topology is up to homeomorphism a quotient topology generated by an equivalence relation $\backsim$.

THEOREM 1.2.29 If $Y$ is a topological space, $X$ is a set, $f : Y \to X$ is surjective and $\backsim$ is the equivalence relation on $Y$ defined by $y \backsim y'$ and only if $f(y) = f(y')$, then $Y/\backsim$ and $X$ are homeomorphic each furnished with its quotient topology.

**Proof.**

The homeomorphism is the bijective map $g : Y/\backsim \to X$ given above. Since $g \circ q = f$, from Proposition 1.2.24, we have that $g$ is continuous. Similarly since $g^{-1} \circ f = q$, we see that $g^{-1}$ is continuous. □

REMARK 1.2.30 Instead of starting with a relation $\backsim$, we might assume that $Y$ is partitioned into a disjoint collection $\mathcal{F}$ of subsets. Then an equivalence relation $\backsim$ is defined by letting $y \backsim y'$ if and only if $y, y'$ belong to the same set member of the collection $\mathcal{F}$. In the case of Example 1.2.28,

$$\mathcal{F} = \{\{0,1\} \text{ and the collection of all singletons other than } \{0\},\ \{1\}\ \}.$$

By Theorem 1.2.29, $Y/\backsim$ is homeomorphic to the unit sphere in $\mathbb{R}^2$.