Лекция 1.6 — "1.6 Пространства отображений"

Введение в нелинейный функциональный анализ

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В этой лекции наметим несколько подходов к введению топологии $C(X, Y)$ всех непрерывных отображений топологического пространства X в топологическое пространство Y, далеко не исчерпав при этом списка возможных топологизаций.

Пусть X — множество, (Y, d) — метрическое пространство, а $B(X, Y)$ — множество таких отображений $f\colon X \to Y$, что

$$diam\, f(X) < \infty.$$

Введем на множестве $B(X, Y)$ метрику равномерной сходимости, положив

$$\rho(f, g) = \sup_{x \in X} \rho(f(x), g(x)).$$

Заметим, что пространство $(B(X, Y), \rho)$ будет полным если (Y, d) является полном метрическим пространством.

The space of all continuous functions from one topological space into another plays an important role in analysis and so it is reasonable to look for useful topologies defined on it.
In particular, we look for a topology which makes continuous the evaluation map

$$(f, x) \to f(x).$$

So let X, Y be topological spaces. By $C(X,Y)$ we denote the space of all continuous functions from X into Y.

DEFINITION 1.6.1 Let $K \subseteq X$ be compact, $U \subseteq Y$ be open and set

$$W(K,U) = \{f \in C(X,Y) \ : \ f(K) \subseteq U\}.$$

The "compact-open topology" (or "c-topology") on $C(X,Y)$ is the topology which has as subbase the family

$$\{W(K,U) \ : \ K \subseteq X \text{ compact}, \ U \subseteq Y \text{ open}\}.$$

Thus every basic element for the compact-open topology is of the form

$$\bigcap_{k=1}^{n} W(K_k, U_k)$$

with $K_k \subseteq X$ compact and $U_k \subseteq Y$ open, $k = 1, \ldots, n$.

Удобно компактно-открытую топологию вводить с помощью π-сетей.

Семейство λ непустых подмножеств топологического пространства X называется π-сетью, если для любого непустого открытого подмножества V пространства X найдется элемент семейства λ, лежащий в множестве U. Будем рассматривать компактные π-сети, т.е. π-сети, элементы которых — компактные множества.

Для π-сети λ топологического пространства X обозначим B_λ семейства всех множеств $W(K, U)$, где $K \in \lambda$, U — произвольное открытое подмножество пространства Y.

Обозначим $C_\lambda(X, Y)$ топологическое пространство, точки которого — элементы $C(X, Y)$, а семейство B_λ составляет предбазу топологии.

Свойства введенной топологии изучается в следующей теореме.

Теорема 1.6.1. Пусть X — компакт, λ — множество всех его замкнутых подмножеств, Y — метрическое пространство. Тогда топология пространства $C_\lambda(X, Y)$ совпадает с топологией равномерной сходимости на множестве $C(X, Y)$ (т.е. с топологией, порожденной метрикой равномерной сходимости).

В описанном подходе нас интересуют два крайних случая.

Предполагаем, что X есть T_1-пространство. Обозначим p семейство всех одноточечных подмножеств пространства X.
Топология пространства $C_p(X, Y)$ называется топологией поточечной сходимости.

Другой крайний случай, рассмотренный авторами книги, когда семейство λ состоит из всех замкнутых компактных подмножеств пространства X. Эта топология называется компактно-открытой топологией, иногда, кратко, компактной топологией. Ее связь с топологией равномерной сходимости показана в теореме 1.6.1.

REMARK 1.6.2 The c-topology is determined by using topological notions on X and Y and so it is uniquely determined by the topologies on X and Y.

Moreover, if $x \in X$ and $U \subseteq Y$ is open, then because $\{x\}$ is compact in X, the set

$$W(\{x\}, U) = \{f \in C(X, Y) \ : \ f(x) \in U\}$$

is c-open. So the topology of pointwise convergence (the relative product topology on $C(X, Y)$ viewed as a subset of Y^X) is weaker than the c-topology.

Finally it is easy to check that

$$\bigcap_{k=1}^{n} W(K_k, U) = W(\bigcup_{k=1}^{n} K_k, U), \quad \bigcap_{k=1}^{n} W(K, U_k) = W(K, \bigcap_{k=1}^{n} U_k),$$

$$\bigcap_{k=1}^{n} W(K_k, U_k) \subseteq W(\bigcup_{k=1}^{n} K_k, \bigcup_{k=1}^{n} U_k), \quad \overline{W(K, U)}^{c} \subseteq W(K, \overline{U}^{c}).$$

Кроме того, полезно использовать и условие замкнутости множеств в $C_\lambda(X, Y)$.

Для любого замкнутого в пространстве Y множества H и любого подмножества M пространства X множество $W(M, H)$ замкнуто в пространстве $C_\lambda(X, Y)$.

Это сразу следует из равенств

$$W(M, H) = \{g : g \in C(X, Y), g(M) \subset H\} =$$

$$= C(X, Y) \setminus \left(\bigcup_{x \in M} W(\{x\}, Y \setminus H) \right).$$

PROPOSITION 1.6.3 If X, Y are topological spaces and $C(X, Y)$ is furnished with the c-topology,

then

(a) $C(X, Y)$ is Hausdorff if and only if Y is Hausdorff;

(b) $C(X, Y)$ is regular if and only if Y is regular.

REMARK 1.6.4 If Y is normal or first countable or second countable, then $C(X, Y)$ with the c-topology need not have the same properties.

Indeed, if X is discrete (i.e. is equipped with the discrete topology), then the only compact sets are finite and so the c-topology coincides with the topology of pointwise convergence, i.e. $C(X, Y)$ with the c-topology is homeomorphic to

$$Y^X = \prod_{x \in X} Y_x$$

where each Y_x is a copy of Y.

Reca11 that the product of normal spaces (resp. of first countable, second countable spaces) may fail to have the corresponding property.

Now let X, Y, Z be three topological spaces.
If $f \in C(X,Y)$ and $g \in C(Y,Z)$, then $g \circ f \in C(X,Z)$.
So we can define the map

$$\xi\colon C(X,Y) \times C(Y,Z) \to C(X,Z)$$

by

$$\xi(f,g) = g \circ f$$

(the composition map).
We equip $C(X,Y)$, $C(Y,Z)$ and $C(X,Z)$ with their respective c-topologies and examine the continuity of ξ.

PROPOSITION 1.6.5 The map $\xi\colon C(X,Y) \times C(Y,Z) \to C(X,Z)$ is continuous in each argument separately.

What about joint continuity of ξ.

PROPOSITION 1.6.6 If X, Z are Hausdorff topological spaces and Y is locally compact,

then

$$\xi\colon C(X,Y)\times C(Y,Z)\to C(X,Z)$$

is continuous.

DEFINITION 1.6.7 The map

$$e\colon C(X,Y)\times X\to Y$$

defined by

$$e(f,x)=f(x)$$

is called the "evaluation map".

Also if we fix $x\in X$, the map

$$e_x\colon C(X,Y)\to Y$$

defined by

$$e_x(f)=f(x)$$

is called the "evaluation at x map".

The evaluation map is important in analysis and so we need to know its continuity properties.
In what follows $C(X, Y)$ is equipped with the c-topology.

PROPOSITION 1.6.8 (a) For each fixed $x \in X$, the evaluation at x map e_x is continuous.

(b) If Y is locally compact, then the evaluation map e is continuous.

Next we want to determine the compact subsets of $C(X, Y)$.
It is clear that such a result has important applications in analysis.
The following notion is crucial in this effort.

DEFINITION 1.6.9 Let X be a topological space and (Y, d) be a metric space.

A family $\mathcal{F}$ of functions from X to Y is said to be "equicontinuous at $x \in X$", if given $\varepsilon > 0$ we can find $U \in \mathcal{N}(x)$ such that

$$d(f(x'), f(x)) < \varepsilon$$

for all $x' \in U$ and all $f \in \mathcal{F}$.

The family $\mathcal{F}$ is said to be "equicontinuous" on X, if it is equicontinuous at each point $x \in X$.

Since we will assume that the range space Y is a metric, we can exploit this fact and define another topology on $C(X, Y)$ as follows.

DEFINITION 1.6.10 Let X be a topological space and $Y, d)$ be a metric space.

Given $f \in C(X, Y), K \subseteq X$ compact and $\varepsilon > 0$, we define

$$B_{K,\varepsilon}(f) = \{g \in C(X, Y) \ : \ \sup\{d(g(x), f(x)) \ : \ x \in K\} < \varepsilon\}.$$

The sets

$$B_{K,\varepsilon}(f)$$

form a basis for a topology on $C(X, Y)$.

This topology is called the "topology of uniform convergence on compacta" (or the "topology of compact convergence").

REMARK 1.6.11 Note that while the compact-open topology and the topology of pointwise convergence
(i.e. the relativization on $C(X, Y)$ of the product topology of Y^X)
are defined without assuming a metric on Y, the above definition explicitly uses the metric of Y.
Nevertheless, we have the following remarkable result.

THEOREM 1.6.12 If X is a topological space and (Y, d) is a metric space,

then

on $C(X, Y)$ the c-topology and the topology of uniform convergence on compacta coincide.

COROLLARY 1.6.13 If X is a topological space and Y is a metric space,

then

the topology of uniform convergence on compacta on $C(X, Y)$ does not depend on the metric of Y.

As we already pointed out in Remark 1.6.2, the topology of pointwise convergence is in general weaker than the c-topology
(or equivalently than the topology of uniform convergence on compacta, see Theorem 1.6.12).
Now we will show that on equicontinuous subsets of $C(X, Y)$ the two topologies coincide.

PROPOSITION 1.6.14 If X is a topological space, (Y, d) is a metric space and $\mathcal{F} \subseteq C(X, Y)$ is equicontinuous,

then

the topology of pointwise convergence and the c-topology coincide on $\mathcal{F}$.

PROPOSITION 1.6.15 If X is a topological space, (Y, d) is a metric space and

$$\mathcal{F} \subseteq C(X, Y)$$

is equicontinuous,

then

$$\overline{\mathcal{F}}^{p}$$

(the closure in the pointwise convergence topology)
is also equicontinuous.

Now we are ready for the theorem characterizing the compact subsets of $C(X, Y)$ with the c-topology.
The theorem that does this is known as the "Arzelà-Ascoli theorem".

THEOREM 1.6.16 If X is a locally compact space, (Y, d) is a metric space and

$$\mathcal{F} \subseteq C(X, Y),$$

then

$\overline{\mathcal{F}}$ (the closure in the c-topology)

is compact

if and only if

$\mathcal{F}$ is equicontinuous and for each $x \in X$,

$$\mathcal{F}(x) = \{f(x) \ : \ f \in \mathcal{F}\}$$

is relatively compact in Y.

A moment's reflection on the second part of the above proof, reveals that the local compactness of X is not needed.
So we have

COROLLARY 1.6.17 If X is a topological space, (Y, d) is a metric space and $\mathcal{F} \subseteq C(X, Y)$ is a subset

such that

(a) $\mathcal{F}$ is equicontinuous on X and

(b) for every $x \in X, \mathcal{F}(x) \subseteq Y$ is relatively compact,

then

$$\overline{\mathcal{F}}$$

is c-compact and equicontinuous on X.

Let us also state the classical version of the Arzelà-Ascoli theorem, which is a direct consequence of Theorem 1.6.16.

COROLLARY 1.6.18 If X is a compact topological space and

$$\mathcal{F} \subseteq C(X, \mathbb{R}^N),$$

then

$$\mathcal{F}$$

is compact for the sup-metric ρ topology

if and only if

it is closed, bounded
(i.e. there exists $M > 0$ such that $\|f(x)\| \leq M$ for all $f \in \mathcal{F}$ and all $x \in X$)
and equicontinuous.

When X is compact and (Y, d) is a metric space, then by Theorem 1.6.12 we know that the c-topology on $C(X, Y)$ can be metrized using the sup-metric ρ defined by

$$\rho(f, g) = \sup\{d(f(x), g(y)) \ : \ x \in X\}$$

Let us investigate a little further the metric space $(C(X, Y), \rho)$.

PROPOSITION 1.6.19 If X is a compact topological space and Y is a metric space,

then

$(C(X, Y), \rho)$ is complete

if and only if

(Y, d) is complete.

A careful reading of the second part of the above proof, leads us to the following standard result of introductory analysis.

PROPOSITION 1.6.20 If X is a topological space, (Y, d) is a metric space,

$$\{f_n\}_{n\geq 1} \subseteq C(X, Y),\ f : X \to Y$$

and

$$\sup\{d(f_n(x), f(x)) \ : \ x \in X\} \to 0$$

as $n \to \infty$
(i.e. f_n converges uniformly to f),

then

f is continuous too
(i.e. $f \in C(X, Y)$).

REMARK 1.6.21 From Corollary 1.6.13 we know that the topology on $C(X, Y)$ induced by the sup-metric ρ, depends only on the topology Y, not on the particular metric d.

So if d_1 and d_2 are compatible metrics on Y, then the corresponding sup-metrics ρ_1 and ρ_2, generate the same topology on $C(X, Y)$.

So we can view $C(X, Y)$ as a topological space without specifying a metric and we can simply refer to the "topology of uniform convergence" on $C(X, Y)$.

PROPOSITION 1.6.22 If X is a compact, metrizable space and Y is a separable, metrizable space,

then

the space $C(X, Y)$ with the topology of uniform convergence is metrizable and separable

(hence so is for the c-topology; Theorem 1.6.12).

In general the pointwise convergence limit of continuous function is not a continuous function. With additional hypotheses we can assert the continuity of the limit function and that the convergence is in fact uniform. This is done in the next theorem, which is known as "Dini's theorem".

THEOREM 1.6.23 If X ia countably compact topological space,

$$\{f_n\}_{n\geq 1}$$

is a decreasing (resp. increasing) sequence of $\mathbb{R}$-valued upper (resp. lower) semicontinuous functions on X and

$$\lim_{n\to\infty} f_n(x) = f(x),$$

where f is lower (resp. upper) semicontinuous,

then

f is continuous and f_n converges to f uniformly (i.e. $\sup_{x\in X} |f_n(x) - f(x)| \to 0$).

All the hypotheses in Theorem 1.6.23 are needed.

EXAMPLE 1.6.24 The functions $x^n \downarrow 0$ on $[0, 1)$ but the convergence is not uniform.

The space $[0, 1)$ is not compact. On the compact space $[0, 1]$, $x^n \to \chi_{\{1\}}$ but again the convergence is not uniform.

The limit function $\chi_{\{1\}}$ is not lower semicontinuous.