Лекция 1.7 — "1.7 Заключительные замечания"

Введение в нелинейный функциональный анализ

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21 марта 2011 г.

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Здесь будет дан краткий очерк развития топологии, а также приведены некоторые сведения, которые не нашли отражения в предыдущих лекциях.

R.I.1 The notion of topological space was developed during the first two decades of the 20^{th} century.
A substantial part of the initial development of the theory can be found in the book of Hausdorff (1914), who introduced the notion of Hausdorff space and gave the definition of continuous functions in terms of open sets. The classic book of Hausdorff marks that moment, where the whole subject of Topology (or Analysis Situs as it was also known) comes of age. Before the appearance of the book of Hausdorff, the knowledge of topology was scattered and diffuse.
Hausdorff did also fundamental work on the foundations of mathematics (set-theory) and on measure theory (Hausdorff measure).

For sets of real numbers, Cantor (1872) defined the notions of neighborhood and of cluster point and also in a subsequent paper (see Cantor (1883)) introduced the notion of closure of a set .

The subbase of a topology τ was introduced by Kelley (1955), p. 48.

Regularity was first used by Vietoris (1921) and Tietze (1923) and normality by Tietze (1923).

Nets were first introduced by Moore-Smith (1922) and for this reason in some old books the theory of convergence of nets is called "Moore-Smith convergence theory". A nice presentation of this theory can be found in the survey article of McShane (1952).

There is an alternative approach to deal with convergence in general topological spaces, using filters and ultrafilters. Filters and ultrafilters were introduced by Cartan (1937) and promoted by Bourbaki (1966).

DEFINITION 1.7.1 Let X be a set and $\mathcal{F} \subseteq 2^X \setminus \{\emptyset\}, \mathcal{F} \neq \emptyset$.
We say that $\mathcal{F}$ is a "filter base in X", if for any $U, V \in \mathcal{F}$ we can find $W \in \mathcal{F}$ such that

$$W \subseteq U \cap V.$$

A filter base $\mathcal{F}$ is said to be a "filter", if $U \in \mathcal{F}$, $U \subseteq V \subseteq X$ imply that $V \in \mathcal{F}$.

Also a filter $\mathcal{F}$ in X is said to be an "ultrafilter", if for all $U \subseteq X$, either $U \in \mathcal{F}$ or $U^c \in \mathcal{F}$.

Finally, a filter base $\mathcal{F}$ is said to "converge to $x \in X$", written $\mathcal{F} \to x$, if $\mathcal{N}(x) \subseteq \mathcal{F}$.

REMARK 1.7.2 If $\mathcal{F}$ is a filter base, then

$$\mathcal{S} = \{V \subseteq X \ : \ U \subseteq V, \ U \in \mathcal{F}\}$$

is a filter and $\mathcal{F}$ is said to be a "base" for $\mathcal{F}$.

The simplest ultrafilters are of the form

$\{U \subseteq X \ : \ x \in U\}, \ x \in X.$ These are called "point ultrafilters".

The existence of non-point ultrafilters depends on the axiom of choice.

Every filter $\mathcal{F}$ is included in some ultrafilter and a filter $\mathcal{F}$ is an ultrafilter if and only if it is maximal with respect to inclusion, i.e. if $\mathcal{F} \subseteq \mathcal{S}$ and $\mathcal{F}$ is a filter, then $\mathcal{F} = \mathcal{S}$.

Note that some filters converging to a point x are included in the point ultrafilter of all sets containing x, but $\mathcal{F} = \{(0, \frac{1}{n}\}$ is a filter base, $\mathcal{F} \to 0$ but no set in the base contains 0.

PROPOSITION 1.7.3 Let (X, τ) be a topological space.

(a) $U \in \tau$ if and only if U belongs to every filter which converge to a point of U;

(b) A point x is an accumulation point of $A \subseteq X$ if and only if $A \backslash \{x\}$ belongs to some filter which converges to x.

(c) If $\mathcal{F}$ is a filter which converges to x and $\mathcal{S}$ is a filter which contains $\mathcal{F}$, then $\mathcal{S} \to x$.

(d) If $(Y, \widehat{\tau})$ is another topological space and $f : X \to Y$, then f is continuous if and only if for every filter base $\mathcal{F} \to x$ we have $f(\mathcal{F}) \to f(x)$.

REMARK 1.7.4 Nets and filters lead to essentially equivalent theories
We can pass from one to the other as follows.

If $\{x_\alpha\}_{\alpha \in D}$ is a net in X, then the family

$$\mathcal{F} = \{A \subseteq X \ : \ \{x_\alpha\}_{\alpha \in D} \text{ is eventually in } A\ \}$$

is a filter in X
(filter generated by the net $\{x_\alpha\}_{\alpha \in D}$).

Conversely let $\mathcal{F}$ be a filter in X and let

$$D = \{(x, A) \ : \ x \in A,\ A \in \mathcal{F}\ \}.$$

D becomes a directed set by setting

$$(x, A) \leq (y, C) \ \text{ if and only if } \ C \subseteq A.$$

REMARK 1.7.4 — завершение

Let $\varphi\colon D \to X$ be defined by $\varphi(x, A) = x$. This is the net generated by the filter $\mathcal{F}$.

We remark that $\mathcal{F}$ is precisely the family of all sets A such that the net

$$\{\varphi(x, A)\}_{(x,A)\in D}$$

is eventually in A.

However we find the convergence theory in terms of filters very unintuitive and think that the theory in terms of nets is mach better and compatible with the classical theory of convergence in terms of sequences.

Semicontinuous functions were introduced by Baire (1899).
Lower semicontinuity became the basic tool in Calculus of Variations by Tonelli (1921).

First countable spaces were introduced by Hausdorff (1914).

Theorem 1.1.51 (Urysohn's lemma), was proved by Urysohn (1925a) and Theorem 1.1.53 (Tietze extension theorem) by Tietze (1915).

Extensions of Theorem 1.1.53 can be found in Dugundji (1951) (see Theorem 4.8.5 in this volume), Hanner (1952) and Dowker (1952).
Urysohn's lemma (Theorem 1.1.51) was published after his premature death (drowned while swimming in the Atlantic, August 1924).

Completely regular topological spaces were introduced by Tychonov (1930) and for this reason are also known as "Tychonov spaces" .

R.I.2 It was Tychonov (1935) who introduced the notion of product topology.
Weak topologies are just a natural "extension" of the product topology.
Detailed discussions of weak topologies can be found in the books of Megginson (1998) and Wilansky (1983).
The quotient topology was introduced by Kelley (1955), p. 94.

The following straight forward observations are useful in many occasions.

PROPOSITION 1.7.5 If X, Y are topological spaces and $A \subseteq X, B \subseteq Y$,

then

(a) $\overline{A \times B} = \overline{A} \times \overline{B}$;

(b) $int\ A \times B = int\ A \times int\ B$;

(c) $bd\ (A \times B) = (bd\ A \cap \overline{B}) \cup (\overline{A} \cap bd\ B)$.

R.I.3 Some books in the definition of compactness also include the Hausdorff property (see Bourbaki (1966), p. 45).

Initially the concept of compactness was approached using sequences and countable covers.

One of the first steps in the direction, was taken by Bernard Bolzano (1781- 1848).

Здесь было бы уместно указать на роль П.С. Александрова и П.С. Урысона в создании теории компактных пространств.

О Больцано

At this point we would like to pause for a moment and say a few words about this remarkable Czech mathematician and philosopher. The official history of mathematics has not treated him fairly. If due credit were given, many theorems and definitions and analysis would have had at least his name too. Bolzano was born in Prague and studied Philosophy, Theology and Mathematics where he got his doctorate (his thesis was in geometry, 1805). He became a priest and was appointed professor of theology at the University of Prague. His progressive (socialist) ideology and his active participation in the struggle of the Czech people for greater autonomy and independence from the Habsburg regime, made him very popular among his countrymen and very unpopular among the ruling class.
So he was fired in 1819 and for the rest of his life was financially supported by friends. For this reason he could not publish his mathematical and philosophical ideas. During this period the situation in mathematics was characterized by instability.

Еще о Больцано

Calculus had become well-known, but there were no precise definitions and not much of the careful reasoning with which we are accustomed today. Bolzano was the first to provide modern rigor. He was extremely prolific. His total output (including mathematical works) is expected to fill 56 volumes. However , his fundamental contribution went unnoticed and were later rediscovered by others, who took all the credit.
One such person is the famous French mathematician Augustin-Louis Cauchy (1789-1857), whose life and political ideas are the complements of those of Bolzano. Cauchy always lived a comfortable life of distinction, he was very conservative (a royalist) and for his devotion to the crown (and of course his mathematical achievements), was made baron by the French king Charles X.
However, a fair account of history should say that Bolzano introduced exactness before Cauchy. After this (we believe justified) tribute to Bolzano, let us return to our historical remarks on compactness.

О Бореле, Александрове и Урысоне

Borel (1895) showed that any open cover of a closed, bounded interval in $\mathbb{R}$, by a sequence of open subintervals, has a finite subcover. In full generality the definition of compactness was given by Alexandrov-Urysohn (1924). They used the name "bicompact space which we still encounter in some Russian books. In terms of filters, compactness can be characterized as follows:

PROPOSITION 1.7.6 A topological space (X, τ) is compact if and only if every ultrafilter in X converges.

Proposition 1.3.8 was proved by Alexandrov (1926).
Theorem 1.3.10 says that any two comparable compact Hausdorff topologies on X are actually equal (see Exercise I.15).
For continuous f, Theorem 1.3.11 was proved by Weierstrass (1894) and in its present form is due to Tonelli (1921).

О Тихонове

Theorem 1.3.23 was proved by Tychonov (1935) and Kelley (1955), p. 143, says that it is "probably the most important single theorem in general topology".

An earlier version of this theorem can be found in Frechet (1906) who deals with a countable product of closed, bounded intervals.

Theorem 1.3.39 is due to Alexandrov and can Ua found in Alexandrov-Hopf (1935,1965), p. 93. While the one-point compactification is easy to describe, it is not satisfactory in one important respect .

The space of continuous functions on the compactified space can be very different from the space of bounded continuous functions on the underlying topological space.

Every continuous $\mathbb{R}$-valued function on X^*, defines a bounded continuous $\mathbb{R}$-valued function on X.

However, not every bounded continuous function on X extends to a continuous function on X^*.

О компактификациях

For example, the sin function can not be extended from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$.
Completely regular spaces (i.e. $T_{3\frac{1}{2}}$-spaces) possess a compactification that avoids this defect . This compactification is known as the Stone-Cech compactification. It was introduced and studied by Cech (1937) and Stone (1937).
Paracompactness was introduced by Dieudonne (1944). For further information on paracompactness we refer to the paper of Michael (1953).
Theorem 1.3.50, which justifies the introduction of paracompact spaces is due to Michael (1953).

От Фреше

R.I.4 The notion of a metric was introduced by Frechet (1906) in his thesis.
The term "metric space" is due to Hausdorff (1914), p. 211.
The notion of complete metric space is also due to Frechet (1906).
Hausdorff (1914), p. 315–316, proved that every metric space has a completion (Proposition 1.4.22).
Here we have reproduced the proof of Hausdorff .
A shorter proof using the space of bounded continuous functions was given by Kuratowski (1935) (see also Dugundji (1966), Theorem XIV.6.1, p. 304).
Theorem 1.4.16 was proved by Cantor (1880).
Hausdorf (1914), p. 311–315, introduced the concept of total boundedness and proved Theorem 1.4.26.
"Polish spaces" were named this way, in recognition of many substantial contributions in the analysis of such spaces made by the Polish mathematicians of the 1920's and 1930's.

Бэр

In Theorem 1.4.33 the necessity part was proved by Mazurkiewicz (1916), while the sufficiency part is due to Alexandrov (1924), with a shorter proof given by Hausdorff (1914).
Dugundji (1966), p. 308, attributes the whole theorem to Mazurkiewicz.
Theorems 1.4.38 and 1.4.39 are known as "Baire category theorem".
The first result in this direction is due to Osgood (1897) who worked with $\mathbb{R}$ and later Baire (1899) in his thesis did the same a thing for $\mathbb{R}^N$.
Theorems 1.4.48 and 1.4.50 are due to Urysohn (1925b).
There is another metrization theorem due to Nagata-Smirnov (see for example Dugundji (1966), Theorem IX.9.1, p. 194).

R.I.5 The theory of uniform spaces began with A. Weil (1937).
Another similar notion used in order to define the concept of uniformly continuous functions is that of proximity space (see for example Kuratowski (1966), p. 230).
In Proposition 1.5.10 if D is not necessarily dense in X and $f\colon D \to \mathbb{R}$ is uniformly continuous and bounded, then it still admits a continuous extension $\hat{f}$ and

$$\sup_{x\in D} |f(x)| = \sup_{x\in D} |\hat{f}(x)|.$$

However we can no longer guarantee the uniqueness of $\hat{f}$.
The notion of connectedness originates in the work of Jordan (1892). The notion of connectedness can be used to define a generalization of the concept of monotone mapping.

DEFINITION 1.7.7 Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous map. We say that f is "monotone", if for each connected $C \subseteq Y$, $f^{-1}(C)$ is connected.

Proposition 1.5.27 was proved by Knaster-Kuratowski (1921).
The notion of component of a topological space was introduced by Hausdorff (1914).
Proposition 1.5.34 is due to Hahn (1921).

DEFINITION 1.7.8 A compact connected Hausdorff space is said to be a "continuum".

A more detailed study of locally connected spaces can be found in Whyburn (1942).

R.1.6 The compact-open topology was defined by Fox (1945). In addition to the results o Proposition 1.6.3, one can also prove:

PROPOSITION 1.7.9 If X, Y are topological spaces and $C(X,Y)$ is furnished with the c-topology, then
(a) if Y is completely regular, so is $C(X,Y)$;
(b) if X, Y are separable metric spaces, $C(X,Y)$ is hereditarily normal, hereditarily Lindelöf and hereditarily separable.

Proposition 1.6.8 was proved by Fox (1945) who also proved that the compact-open topology is smaller than every topology on $C(X,Y)$ which makes the evaluation map e jointly continuous.
Arens (1946) proved that in general (i.e. if Y is not locally compact) there is no smallest topology for which the evaluation map is jointly continuous.
The notion of equicontinuity is due to Arzela (1882-1883) and Ascoli (1883-1884).

Продолжение

The Arzela-Ascoli theorem was first proved for $C([0,1])$ by Arzela, (1882-1883) (the necessary part) and by Ascoli (1883-1884) (the sufficient part).
The general form presented in Theorem 1.6.16 is due to Kelley (1955) , p. 233–234.
In the beginning of the 19-th century (1821) Cauchy claimed that the limit of continuous functions is a continuous function.
The fallacy of this claim was pointed out by Abel (1826) in 1826.
Later Stokes (1847-1848), von Seidel (1847-1849) and Cauchy (1900) independently realized that uniform convergence is sufficient to guarantee the continuity of the limit of a sequence of continuous functions.
Moreover, in 1878 Dini gave necessary and sufficient conditions for continuity of the limit function at a point and also stated Theorem 1.6.23 (with $X = [0,1]$ and the sequence $\{f_n\}_{n\geq 1}$ consisting of continuous functions).

Окончание

The sup-metric topology was introduced by Frechet (1906), who also says that the first mathematician to make systematic use of uniform convergence was Weierstrass.
There are several well- written books on point-set topology. In writing this chapter we consulted the following ones:
Bourbaki (1966) , Buskes-van Rooij (1997), Kelley (1955), Kuratowski (1966), (1968), Munkres (1975), Nagata (1968), Whyburn (1942) and Wilansky (1983).
For examples and counterexamples of a rich variety of properties, we refer to the book by Steen-Seebach (1970).
A well-written survey of the point-set topology with a treasure of references, can be found in the article of Alexandrov-Fedorchuk (1978).