Лекция Т.2.1 — "2.1 Меры и измеримые функции"

Введение в нелинейный функциональный анализ

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Понятие меры, возникшее первоначально в теории функций действительного переменного, в настоящее время играет первостепенную роль в самых разнообразных отделах математики. Наряду с теорией функций действительного переменного понятием меры, в той или иной форме, широко пользуются теория вероятностей, функциональный анализ, топологическая алгебра, качественная теория дифференциальных уравнений и т. п. Различные отделы теоретической физики, используя методы теории вероятностей, функционального анализа, эргодические теоремы, также оказываются связанными в известной степени с понятием меры. История интегрирования восходит к Архимеду, Ньютону и Лейбницу. Однако строгое обоснование теории интегрирования стало возможно только во второй половине XIX-го века, благодаря Коши, Дирихле и Риману. Кульминацией этого подхода стали работы Лебега, появившиеся в 1902-1903, в которых он определил понятие меры и измеримой функции, ныне общепринятое. Современная теория меры строится аксиоматически, опираясь на базовые факты теории множеств, топологии и анализа.

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2. Рохлин В. А. Об основных понятиях теории меры // Матем. сб. 25:1 (1949), 107-150 (Рохлин.djvu).

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The theory of measures is an essential ingredient of integration theory and deals with set functions (called measures), defined on certain collection of sets which are different from what we experienced in topology. Some properties of measures require that the collection of sets, on which measures are defined, are to be closed with respect to countable set operations. This leads to our first abstract definition.

Определение поля или алгебры множеств

DEFINITION 2.1.1 Let Ω be any set and $\mathcal{S}$ be a nonempty family of subsets of Ω. We say that $\mathcal{S}$ is a "field" (or "algebra") of sets, if

(a) $\emptyset \in \mathcal{S}$;

(b) if $\{A_k\}_{k=1}^N \subseteq \mathcal{S}$, then $\bigcup_{k=1}^N A_k \in \mathcal{S}$;

(c) if $A \in \mathcal{S}$, then $\Omega \backslash A = A^c \in \mathcal{S}$.

REMARK 2.1.2 According to this definition a field is closed under finite unions and complements.

From these two facts and De Morgan's law, it follows that a field is also closed under finite intersections.

Moreover, since $\emptyset \in \mathcal{S}$, from property (c) we have that $\Omega \in \mathcal{S}$.

EXAMPLE 2.1.3 Let $\Omega = (0, 1]$

and let

$\mathcal{S}$ consist of $\emptyset$ and all finite unions of half-open intervals $(a, b]$ contained in Ω.

Then $\mathcal{S}$ is a field.

A field is closed under finite set operations. However in many instances in order to check various properties, we need to perform countable set operations. This leads to the next definition. With this definition we can develop a coherent theory of measure and integration.

DEFINITION 2.1.4 Let Ω be a set and Σ a family of subsets of Ω. We say that Σ is a "σ-field" (or a "σ-algebra"), if Σ is a field and it is closed under countable unions, i.e. if

$$\{A_k\}_{k\geq 1} \subseteq \Sigma, \text{ then } \bigcup_{k\geq 1} A_k \in \Sigma.$$

The sets of Σ are called "measurable" or "Σ-measurable".
If $A \subseteq \Omega$ (not necessarily in Σ), we set

$$\Sigma_A = \{A \cap C \;:\; C \in \Sigma\}.$$

Then Σ_A is a field of subsets of A and it is called the "trace σ-field on A" or the "relative σ-field on A".

REMARK 2.1.5 It is easy to see that the intersection of an arbitrary nonempty family of σ-fields of Ω is again a σ-field.

So given $\mathcal{F}$ a family of subsets, we can speak about the smallest σ-field on Ω which contains $\mathcal{F}$ (the intersection of all σ-fields that contain $\mathcal{F}$).

This σ-field is called the "σ-field generated by $\mathcal{F}$" and is denoted by $\sigma(\mathcal{F})$.

This leads to the next definition which is the first link between topology and measure theory.

DEFINITION 2.1.6 If (X, τ) is a topological space,

then

$\sigma(\tau)$ (being the σ-field generated by the topology) is called the "Borel σ-field of X" and is denoted by $\mathcal{B}(X)$.

Sets in $\mathcal{B}(X)$ are called "Borel sets".

Evidently $\mathcal{B}(X)$ contains all sets of type F_σ and of type G_δ, but it also contains many other sets.

Of special interest are the Borel σ-fields $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^N)$.

PROPOSITION 2.1.7 $\mathcal{B}(\mathbb{R})$ is generated by anyone of the following families of sets:

(a) the family of closed sets;

(b) the family of all intervals of the form $(-\infty, b]$;

(c) the family of all intervals of the form $(a, b]$.

In a similar fashion we can describe $\mathcal{B}(\mathbb{R}^N)$.

PROPOSITION 2.1.8 $\mathcal{B}(\mathbb{R}^N)$ is generated by anyone of the following families of sets:

(a) the family of all closed subsets of $\mathbb{R}^N$;

(b) the family of all closed half-spaces of $\mathbb{R}^N$ of the form

$$\{(x_k)_{k=1}^N \in \mathbb{R}^N \ : \ x_k \leq b\}$$

for some index k and some $b \in \mathbb{R}$; and

(c) the family of rectangles of $\mathbb{R}^N$ of the form

$$\{(x_k)_{k=1}^N \ : \ a_k < x_k \leq b_k \text{ for } k = 1, \ldots, N\}.$$

We want to be able to recognize when a field is actually a σ-field. This is done by the next proposition.

PROPOSITION 2.1.9 If Ω is a set and $\mathcal{S}$ is a field on Ω, then $\mathcal{S}$ is a σ-field if one of the following properties holds:

(a) if $\{A_n\}_{n\geq 1} \subseteq \mathcal{S}$ is increasing, then $\bigcup_{n\geq 1} A_n \in \mathcal{S}$; or

(b) if $\{A_n\}_{n\geq 1} \subseteq \mathcal{S}$ is decreasing, then $\bigcap_{n\geq 1} A_n \in \mathcal{S}$.

REMARK 2.1.10 Families that satisfy (a) and (b) of Proposition 2.1.9 are called "monotone classes".

Proposition 2.1.9 says that the smallest monotone class and the smallest σ-field over a field coincide.

Moreover, a monotone class which is also a field, is a σ-field.

DEFINITION 2.1.11 Let Ω be a set and Σ a σ-field on Ω. A set function $\mu\colon \Sigma \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is said to be a "measure" (or "countably additive set function" or "σ-additive set function"), if $\mu(\emptyset) = 0$ and whenever $\{A_n\}_{n\geq 1} \subseteq \Sigma$ are pairwise disjoint sets, then

$$\mu(\bigcup_{n\geq 1} A_n) = \sum_{n\geq 1} \mu(A_n).$$

If $\mathcal{S}$ is a field on Ω and $\mu\colon \mathcal{S} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is a set-function such that $\mu(\emptyset) = 0$ and for every family $\{A_k\}_{k\geq 1}^N \subseteq \Sigma$ of mutually disjoint sets we have

$$\mu(\bigcup_{n\geq 1}^{N} A_k) = \sum_{n\geq 1}^{N} \mu(A_k),$$

then we say that μ is an "additive" (or "finitely additive") set function. If Ω a set and Σ is a σ-field, the pair (Ω, Σ) is said to be a "measurable space". If μ is a measure on Σ, then the triple (Ω, Σ, μ) is called a "measure space". Moreover, if $\mu(\Omega) = 1$, then the triple (Ω, Σ, μ) is called a "probability space" and μ is a "probability measure".

Measures are monotonic and countably subadditive.

PROPOSITION 2.1.12 If (Ω, Σ, μ) is a measure space, then

(a) if $A, B \in \Sigma$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and if in addition $\mu(A) < \infty$, we have that $\mu(B \backslash A) = \mu(B) - \mu(A)$;

(b) if $\{A_k\}_{k \geq 1} \subseteq \Sigma$, then

$$\mu(\bigcup_{k \geq 1} A_k) \leq \sum_{k \geq 1} \mu(A_k)$$

(countable subadditivity).

For monotone sequences we have easily some limit results.

PROPOSITION 2.1.13 If (Ω, Σ, μ) is a measurable space and $\{A_n\}_{n\geq 1} \subseteq \Sigma$, then

(a) if $A_1 \subseteq A_2 \subseteq \ldots$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n\geq 1} A_n);$$

(b) if $A_1 \supseteq A_2 \supseteq \ldots$ and for some $k \geq 1, \mu(A_k) < \infty$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n\geq 1} A_n)$$

We can improve this proposition 2.1.13, if we use the following two set theoretic limits.

DEFINITION 2.1.14 Let $\{A_n\}_{n\geq 1}$ be a sequence of subsets of a set Ω.

We define

$$\liminf_{n\to\infty} A_n = \bigcup_{m\geq 1} \bigcap_{n\geq m} A_n \quad \text{and} \quad \limsup_{n\to\infty} A_n = \bigcap_{m\geq 1} \bigcup_{n\geq m} A_n.$$

If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$ we say that the sequence $\{A_n\}_{n\geq 1}$ converges (set-theoretic) to A and denote it by

$$A = \lim_{n\to\infty} A_n.$$

REMARK 2.1.15 A moments's reflection on the above definition, reveals that

$$\liminf_{n\to\infty} A_n = \{\omega \in \Omega \ : \ \omega \in A_n \ \text{ for all } \ n \geq n_0\}$$

and

$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega \ : \ \omega \in A_n \ \text{ for infinitely many } \ n \geq 1\}.$$

If the sequence $\{A_n\}_{n\geq 1}$ is increasing (resp. decreasing) , then

$$\lim_{n\to\infty} A_n = \bigcup_{n\geq 1} A_n \ \ (\textit{resp.} \ \lim_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n).$$

Finally, if $\{A_n\}_{n\geq 1} \subseteq \Sigma$ with Σ a σ-field, it is clear that $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ belong to Σ.

PROPOSITION 2.1.16 If (Ω, Σ, μ) is a measure space and $\{A_n\}_{n\geq 1} \subseteq \Sigma$, then

$$(a)\ \mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n);$$

(b) if $\mu(\bigcup_{n\geq 1} A_n) < \infty$, then

$$\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n);$$

(c) if $\mu(\bigcup_{n\geq 1} A_n) < \infty$ and $A = \lim_{n\to\infty} A_n$, then

$$\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

The next result is used quite often in analysis and especially in probability theory and is known as the "Borel-Cantelli lemma". First we need a concept from probability theory.

DEFINITION 2.1.17 Let (Ω, Σ, μ) be a probability space.

Let I be an arbitrary index set and let $\{A_i\}_{i \in I}$ be a family of Σ-sets (events in the language of probability theory).

We say that the A_i are independent, if for all finite collections of distinct indices $\{i_1, \ldots, i_n\} \subseteq I$, we have

$$\mu(\bigcap_{k=1}^{n} A_{i_k}) = \prod_{k=1}^{n} \mu(A_k).$$

REMARK 2.1.18 If the sets (events) $\{A_i\}_{i \in I}$ are independent and any set (event) is replaced by its complement, independence is maintained.

PROPOSITION 2.1.19 (a) If (Ω, Σ, μ) is a measure space, $\{A_n\}_{n\geq 1} \subseteq \Sigma$ and $\sum_{n\geq 1} \mu(A_n) < \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 0$.

(b) If (Ω, Σ, μ) is a probability space, $\{A_n\}_{n\geq 1} \subseteq \Sigma$ are independent and $\sum_{n\geq 1} \mu(A_n) = \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 1$.

Now we will develop one of the standard techniques for constructing measures. This involves the notion of outer measure.

DEFINITION 2.1.20 Let Ω be a set and 2^{Ω} be the family of all subsets of Ω.

A set function $\mu^*\colon 2^{\Omega} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is said to be an ''outer measure'', if it satisfies

(a) $\mu^*(\emptyset) = 0$;

(b) if $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$ (monotonicity property); and

(c) if $\{A_n\}_{n\geq 1} \subseteq 2^{\Omega}$, then

$$\mu^*(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \mu^*(A_n)$$

(countable subadditivity property).

Given an outer measure μ^* on Ω, we will show that there is a σ-field Σ_{μ^*} on Ω such that μ^* restricted to Σ_{μ^*} is a measure.

DEFINITION 2.1.21 Let μ^* be an outer measure on a set Ω.

A set $A \subseteq \Omega$ is "μ^*-measurable", if

$$\mu^*(C) = \mu^*(C \cap A) + \mu^*(C \setminus A) = \mu^*(C \cap A) + \mu^*(C \cap A^c)$$

for every $C \in 2^{\Omega}$.

We denote the collection of all μ^*-measurable subsets of Ω by Σ_{μ^*}.

This definition involves an additivity requirement but not any kind of σ-additivity. For this reason the next theorem is rather surprising.

THEOREM 2.1.22 If Ω is a set, μ^* is an outer measure and Σ_{μ^*} the family of all μ^*-measurable subsets of Ω,

then Σ_{μ^*} is a σ-field and $\mu = \mu^*|_{\Sigma_{\mu^*}}$ is a measure.

Now we will use Theorem 2.1.22 to construct the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^N$, $N > 1$.
For $\mathbb{R}$ the primitive notion is that of length of an interval, while in $\mathbb{R}^N$, $N > 1$ the primitive notion is that of volume of an N-dimensional rectangle.

DEFINITION 2.1.23 Let $A \subseteq \mathbb{R}$ and let $\mathcal{C}_A$ be the collection of all sequences $\{I_n\}_{n\geq 1}$ of open intervals such that $A \subseteq \bigcup_{n\geq 1} I_n$.

The "Lebesgue outer measure" $\lambda^* : 2^{\mathbb{R}} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\lambda^*(A) = \inf \left\{ \sum_{n\geq 1} |I_n| \ : \ \{I_n\}_{n\geq 1} \subseteq \mathcal{C}_A \right\}$$

where by $|I_n|$ we denote the length of I_n.

Next we show that λ^* is indeed an outer measure in the sense of Definition 2.1.21.

PROPOSITION 2.1.24 $\lambda^*\colon 2^{\mathbb{R}} \to \overline{\mathbb{R}}_+$ is an outer measure in the sense of Definition 2.1.21 and assigns to each interval of $\mathbb{R}$ its length.

In a similar fashion we can define the Lebesgue outer measure for $\mathbb{R}^N$, $N > 1$.

DEFINITION 2.1.25 Let $A \subseteq \mathbb{R}^N$ and let $\mathcal{C}_A$ be the collection of all sequence $\{R_n\}_{n\geq 1}$ of open N-dimensional rectangles $R_n = \prod_{k=1}^{N} I_{nk}$ with I_{nk} open interval in $\mathbb{R}$ such that $A \subseteq \bigcup_{n\geq 1} R_n$.

The "N-dimensional Lebesgue outer measure" $\lambda_N^* \colon 2^{\mathbb{R}^N} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\lambda_N^*(A) = \inf \left\{ \sum_{n\geq 1} v(R_n) \ : \ \{R_n\}_{n\geq 1} \subseteq \mathcal{C}_A \right\}$$

where by $v(R_n)$ we denote the "volume" of R_n, i.e.

$$v(R_n) = \prod_{k=1}^{N} |I_{nk}|.$$

The following analog of Proposition 2.1.24 holds.

PROPOSITION 2.1.26 $\lambda_N^*\colon 2^{\mathbb{R}^N} \to \overline{\mathbb{R}}_+$ is an outer measure in the sense of Definition 2.1.26 and it assigns to each N-dimensional rectangle

$$R = \prod_{k=1}^{N} I_k,\ \ I_k \subseteq \mathbb{R}$$

interval, its volume

$$v(R) = \prod_{k=1}^{N} |I_k|.$$

Since λ^* and λ_N^* are outer measures, because of Theorem 2.1.22, the following definition makes sense.

DEFINITION 2.1.27 The σ-field Σ_{λ^*} of all λ^*-measurable sets of $\mathbb{R}$, is called the "Lebesgue σ-field of $\mathbb{R}$" and is denoted by $\mathcal{L}(\mathbb{R})$ or simply $\mathcal{L}$.

Similarly $\Sigma_{\lambda_N^*}$ is the "Lebesgue σ-field of $\mathbb{R}^N$" and is denoted by $\mathcal{L}(\mathbb{R}^N)$ or simply $\mathcal{L}_N$.

The sets in $\mathcal{L}$ and $\mathcal{L}_N$ are called "Lebesgue measurable sets".

Also $\lambda = \lambda^*|_{\mathcal{L}}$ is called the "Lebesgue measure on $\mathbb{R}$" and $\lambda_N = \lambda_N^*|_{\mathcal{L}_N}$ is called the "N-dimensional Lebesgue measure".

PROPOSITION 2.1.28 Every Borel set of $\mathbb{R}^N$, $N > 1$, is Lebesgue measurable.

The next proposition shows that Σ_{μ^*} has the property that every subset with μ^*-outer measure zero (usually called μ^*-null set) belongs to Σ_{μ^*}.

PROPOSITION 2.1.29 If Ω is a set, μ^* is an outer measure on Ω and $A \subseteq \Omega$ satisfies $\mu^*(A) = 0$,

then $A \in \Sigma_{\mu^*}$.

In general for a measure space (Ω, Σ, μ) it is not true that subsets of zero measure sets (usually called μ-null sets) necessarily belong to Σ. In order to be able to produce interesting counterexamples and better understand the measure spaces that have this property, we need to introduce and discuss the so-called "Cantor-type sets".

EXAMPLE 2.1.30 "Cantor-ternary sets": Let $I = [0, 1]$. Divide I into three equal parts and remove the open middle third, that is, the interval $A_{11} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Then divide the remaining closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ into three equal parts each and remove the open middle third, namely the intervals $A_{21} = \left(\frac{1}{3^2}, \frac{2}{3^2}\right)$ and $A_{22} = \left(\frac{7}{3^2}, \frac{8}{3^2}\right)$.

Продолжение примера 2.1.30

Then we divide the remaining four closed intervals into three equal parts and remove in each case the open middle third, namely the intervals

$$A_{31} = \left(\frac{1}{3^3}, \frac{2}{3^3}\right), \; A_{32} = \left(\frac{7}{3^3}, \frac{8}{3^3}\right), \; A_{33} = \left(\frac{19}{3^3}, \frac{20}{3^3}\right), \; A_{34} = \left(\frac{25}{3^3}, \frac{26}{3^3}\right).$$

We continue this way ad infinitum. At the n^{th}-step we remove 2^{n-1}"open intervals $\{A_{nk}\}_{k=1}^{2^{n-1}}$ with $|A_{nk}| = \frac{1}{3^n}$. Then the Cantor ternary set is defined by

$$C = I \setminus \bigcup_{n \geq 1} A_n, \text{ where } A_n = \bigcup_{k=1}^{2^{n-1}} A_{nk}.$$

Evidently C is closed and nonempty.

Еще продолжение примера 2.1.30

Indeed the endpoints of the various middle thirds were not removed, so they remain in C and since C is closed so do all the limit points of these endpoints. For example, if we start from $\frac{1}{3}$ and take the closest endpoint in the second step, then this is

$$\frac{1}{3} - \frac{1}{9} = \frac{2}{9}.$$

In the third step the closest endpoint to $\frac{2}{9}$ is

$$\left(\frac{1}{3} - \frac{1}{9} + \frac{2}{27}\right) = \frac{8}{27}$$

and so on. We form a sequence which is convergent to

$$\frac{1}{4} = \sum_{n\geq 1}(-1)^{n+1}\frac{1}{3^n}.$$

Thus there are points in C which are not endpoints of removed open middle thirds.

Это еще пример 2.1.30

Note that the total length of the intervals removed is equal to

$$\frac{1}{3}\sum_{n\geq 1}\left(\frac{2}{3}\right)^{n} = 1.$$

Hence $\lambda(C) = 0$ (i.e C is Lebesgue-null set) and contains no interval, thus it is nowhere dense.
Also C is perfect (i.e. every point is a limit point). Indeed, it suffices to check that every endpoint is a limit point.
But in any neighborhood of any endpoint, there will always be a small interval that is not removed at some step and this interval will contain an endpoint belonging to a subsequent step.

Финал примера 2.1.30

Finally it is easy to check that the map which assigns to each sequence $\{a_n\}_{n\geq 1}$ of 0 and 1 the number

$$\sum_{n\geq 1} \frac{2}{3^n} a_n$$

is a bijection of all such sequences onto C.
Since the set of all sequences with 0 and 1 is uncountable, it follows that the set C has the cardinality of the continuum (i.e. is uncountable).

So recapitulating we have that:

C is a closed, nowhere dense, perfect set which is uncountable and Lebesgue-null. The set C is called the "Cantor ternary set".

REMARK 2.1.31 Another way to see the Cantor ternary set is the following. Every $x \in [0,1]$ has a base 3 (ternary) representation denoted by $x = ._3 a_1 a_2 a_3 \ldots$,where each a_n, $n \geq 1$, is either 0, 1 or 2.
However a representation of this type is not unique.
For example we have

$$\frac{1}{3} = ._3 1000\ldots = ._3 0222\ldots$$

Consider the first interval removed from $[0,1]$ in the construction of C, namely $A_{11} = \left(\frac{1}{3}, \frac{2}{3}\right)$. If $x \in A_{11}$ and $x = ._3 a_1 a_2 a_3 \ldots$, then $a_1 = 1$. Each of the endpoints of A_{11} has two representations:

$$\frac{1}{3} = \frac{1}{3} = ._3 1000\ldots = ._3 0222\ldots \quad \text{and} \quad \frac{2}{3} = ._3 1222\ldots = ._3 2000\ldots$$

No remaining point of $[0,1]$ can have 1 in the first position of its ternary representation. So in the first step of the construction of C, we remove a1l these points $x = ._3 a_1 a_2 a_3 \ldots$, where $a_1 = 1$ and only those points.

Окончание примера 2.1.31

Similarly in the second step of the construction of C we remove those points for which $a_2 = 1$ and only those points are removed. At the i n^{th}-step of the construction, we remove those points and only those points for which $a_n = 1$. At the end C must consist of those points which have ternary representations that contain only the digits 0 and 2, i.e.

$$C = \{x \in [0,1] \ : \ x = {}_{.3}a_1 a_2 a_3 \ldots = \sum_{n\geq 1} \frac{a_n}{3^n} \ \text{with} \ \ a_n = 0 \text{ or } 2 \ \ n \geq 1 \ \}.$$

Two properties of C worth mentioning are the following:

(a) the transformation $x \to 1 - x$ leaves the C and C^c invariant; and

(b) given any $u \in [0,1]$ we can find $x, y \in C$ such that $u = x - y$, i.e. $[0,1] = C - C$.

The next example shows the difference between Lebesgue-null sets and sets of the first category, i.e. between measure theoretically small sets and topologically small sets.
We can have a closed, nowhere dense, perfect set (a Cantor-type set) with relatively large measure.

EXAMPLE 2.1.32 "Cantor-type sets": Choose any sequence of positive numbers $\{\eta_k\}_{k\geq 1}$ such that

$$\ldots < 2^k\eta_k < \ldots 4\eta_2 < 2\eta_1 < \eta_0 = 1.$$

We remove from $[0,1]$ the open interval I_{11} with center $\frac{1}{2}$ and length $1-2\eta_1$, leaving two closed intervals J_{11} and J_{12} each with length η_1. Now from each of the closed intervals J_{11} and J_{12} remove open intervals I_{11} and I_{12} respectively with length $\eta_1 - 2\eta 2$ leaving four closed intervals J_{21}, J_{22}, J_{23} and J_{24} each of length η_2. We continue this ad infinitum. At the n^{th}-step, remain 2^n closed disjoint intervals, each of length η_n.

Продолжение примера 2.1.32

Let

$$\widehat{C}_n = \bigcup_{k=1}^{2^n} J_{nk} \quad \text{and} \quad U_n = \bigcup_{k=1}^{2^{n-1}} I_{nk}.$$

Set

$$\widehat{C} = \bigcap_{n\geq 1} \widehat{C}_n = [0,1] \cap \left(\bigcup_{n\geq 1} U_n \right)^c$$

Then $\lambda(\widehat{C}) = \lim_{n\to\infty} 2^n \eta_n$. So if $\theta \in (0,1)$ and we choose

$$2^n \eta_n = \frac{\theta n + 1}{n+1}$$

then $\lambda(\widehat{C}) = \theta$. If $\eta_n = \frac{1}{3^n}$, then $\widehat{C} = C$ the Cantor ternary set. Note that since no $\widehat{C}_n$ contains an interval of length $\geq \frac{1}{2^n}$, we infer that $\widehat{C}$ contains no interval, hence is nowhere dense. Also if $x \in \widehat{C}$, then $x \in \widehat{C}_n$ for all $n \geq 1$. So there exist $k_n \geq 1$ such that $x \in J_{nk_n}$.

Окончание примера 2.1.32

Given $\varepsilon > 0$ we can find $n \geq 1$ such that $\frac{1}{2^n} < \varepsilon$ and so the endpoints of J_{nk_n} are both in $(x - \varepsilon, x + \varepsilon)$. But these endpoints are in $\widehat{C}$. Hence x is a limit point of $\widehat{C}$ and so $\widehat{C}$ is perfect and of course closed.
So we have that $\widehat{C}$ is closed, nowhere dense, perfect and $\lambda(\widehat{C}) = \theta$, where $\theta \in (0, 1)$.

The set $\widehat{C}$ is said to be a "Cantor-type set".

Moreover, since in a complete metric space a perfect set has cardinality bigger or equal to that of the continuum (see for example Hewitt-Stromberg (1975), p. 72) we have that the cardinality of $\widehat{C}$ is that of the continuum, i.e. $\widehat{C}$ is uncountable.

In Proposition 2.1.29 we saw that Σ_{μ^*} contains all μ^*-null sets and we also mentioned that this is not the case with general measure space.

EXAMPLE 2.1.33 Consider the measure space (Ω, Σ, μ) with $\Omega = [0, 1]$, $\Sigma = \mathcal{L}([0, 1])$ = the Lebesgue σ-field of $[0, 1]$ and λ = the Lebesgue measure on $[0, 1]$. Let C be the Cantor ternary set (Example 2.1.30). Since $\lambda(C) = 0$ and C is uncountable, it follows that there are 2^c Lebesgue-null sets, where c is the cardinality of the continuum. On the other hand if $\Sigma = \mathcal{B}([0, 1])$ = the Borel σ-field of $[0, 1]$, then because $[0, 1]$ is second countable, $\mathcal{B}([0, 1])$ has the cardinality of the continuum, i.e. $card\, \mathcal{B}([0, 1]) = c$. Therefore there exist Lebesgue-null sets which are not Borel sets. So $(\Omega, \mathcal{B}([0, 1]), \lambda)$ is not "complete" according to the next definition.

DEFINITION 2.1.34 Let (Ω, Σ, μ) be a measure space.

The measure μ is said to be "complete", if the conditions $A \in \Sigma, \mu(A) = 0$ and $B \subseteq A$ imply that $B \in \Sigma$.

Then (Ω, Σ, μ) is said to be a "complete measure space".

REMARK 2.1.35 Properly speaking completeness is a property of the σ-field, but it is common practice to use the term complete for the measure.

In what follows a set $B \in \Omega$ is said to be μ-null (or μ-negligible), if there is $A \in \Sigma$ such that $B \subseteq A$ and $\mu(A) = 0$.

Thus a measure μ is complete if and only if every μ-null set belongs to Σ.

It is often convenient to deal with arbitrary subsets of sets of measure zero. In these cases it is useful to have complete measure space. Fortunately, every measure space can be "completed".

PROPOSITION 2.1.36 If (Ω, Σ, μ) is a measure space, $\mathcal{N} = \{Z \ : \ \text{there exists } N \in \Sigma \text{ such that } Z \subseteq N \text{ and } \mu(N) = 0\}$, $\overline{\Sigma} = \{A \cup Z \ : \ A \in \Sigma, Z \in \mathcal{N}\}$ and $\overline{\mu} \colon \overline{\Sigma} \to \overline{\mathbb{R}}_+$ is defined by $\overline{\mu}(A \cup Z) = \mu(A)$, then

(a) $\overline{\Sigma}$ is a σ-field and $\Sigma, \mathcal{N} \subseteq \overline{\Sigma}$;

(b) $\overline{\mu}$ is a measure and $\overline{\mu}|_{\Sigma} = \mu$;

(c) $\overline{\mu}$ is complete ($\overline{\mu}$ is called the completion of μ).

Now we will show that for all $N \geq 1$, $\mathcal{L}_N = \overline{\mathcal{B}(\mathbb{R}^N)}$ for the Lebesgue measure λ_N (when $N = 1$, $\mathcal{L}_1 = \mathcal{L}$ and $\lambda_1 = \lambda$).
We will need two auxiliary results.

LEMMA 2.1.37 If $A \in \mathcal{L}_N, N \geq 1,$

then $\lambda_N(A) = \inf\{\lambda_N(U) \;:\; U \text{ is open}, \; A \subseteq U\} = \sup\{\lambda_N(K) \;:\; K \text{ is compact}, K \subseteq A\}$.

REMARK 2.1.38 Lemma 2.1.37 implies that the Lebesgue measure λ_N is ”regular” (see Definition 2.5.7).

However, we delay the definition and discussion of regularity until Section 2.5, when we will discuss measure theory in conjunction with topology.

LEMMA 2.1.39 If $A \in \mathcal{L}_N$ $(N \geq 1)$,

then there exist $B_1, B_2 \in \mathcal{B}(\mathbb{R}^N)$ such that $B_1 \subseteq A \subseteq B_2$

and $\lambda_N(B_2 \backslash B_1) = 0$.

Using Lemmata 2.1.37 and 2.1.39, we can prove the following proposition (as before $\mathcal{L}_1 = \mathcal{L}$ and $\lambda_1 = \lambda$).

PROPOSITION 2.1.40 The Lebesgue measure λ_N on $(\mathbb{R}^N, \mathcal{L}_N)$ is the completion of the Lebesgue measure $\overset{\circ}{\lambda}_N$ on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$.

In the proofs of the two lemmata and of Proposition 2.1.40, crucial was the fact that $\mathbb{R}^N$ can be expressed as the union of a sequence of Lebesgue measurable sets of finite Lebesgue measure. We formalize this in the next definition.

DEFINITION 2.1.41 Let μ be a measure on a measurable space (Ω, Σ). Then μ is "finite", if $\mu(\Omega) < +\infty$ and it is "σ-finite", if $\Omega = \bigcup_{n\geq 1} \Omega_n, \Omega_n \in \Sigma$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$.

More generally, a set $A \in \Sigma$ is σ-finite under μ, if $A = \bigcup_{n\geq 1} A_n$ with $A_n \in \Sigma$ and $\mu(A_n) < +\infty$ for all $n \geq 1$.

The measure space (Ω, Σ, μ) is said to be "finite" (respectively, "σ-finite"), if μ is finite (respectively, σ-finite).

In general for a given set-function additivity is easily established, but σ-additivity is more difficult to prove.
So we need conditions which will pass us from additivity to σ-additivity.

PROPOSITION 2.1.42 If $\mathcal{F}$ is a field and $\mu\colon \mathcal{F} \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is an additive set function, then

(a) if μ is continuous from below at each $A \in \mathcal{F}$, i.e. if $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ is increasing and $\cup_{n\geq 1} A_n = A$, we have $\mu(A_n) \to \mu(A)$, then it follows that μ is σ-additive;

(b) μ is $\mathbb{R}$-valued and continuous from above at the empty set, i.e. if $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ is decreasing and $\cap_{n\geq 1} A_n = \emptyset$, we have $\mu(A_n) \to 0$, then it follows that μ is σ-additive.

With the next theorem we show that any σ-finite measure on a field $\mathcal{F}$ has a unique extension to the minimal σ-field $\Sigma = \sigma(\mathcal{F})$. For a proof of this fundamental extension theorem (known as "Caratheodory Extension Theorem"), we refer to Dudley (1989), p. 66 or Royden (1968), p. 257.

THEOREM 2.1.43 If μ is a measure on the field $\mathcal{F}$ of subsets of Ω and μ is σ-finite on $\mathcal{F}$ (i.e. $\Omega = \cup_{n\geq 1}\Omega_n$ with $\Omega_n \in \Sigma$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$),

then μ has a unique extension to a measure on $\sigma(\mathcal{F})$.

REMARK 2.1.44 An arbitrary measure μ on $\mathcal{F}$ still has an extension on $\sigma(\mathcal{F})$ but this extension is not necessarily unique.

Concerning the relation of the values of a σ-finite measure μ on a field $\mathcal{F}$ and on the minimal σ-field $\sigma(\mathcal{F})$ generated by $\mathcal{F}$, we have the following result.

PROPOSITION 2.1.45 If (Ω, Σ, μ) is a measure space, $\mathcal{F}$ is a field of subsets of Ω such hat $\Sigma = \sigma(\mathcal{F})$, μ is σ-finite on $\mathcal{F}$ (i.e. $\Omega = \cup_{n \geq 1} \Omega_n$ with $\Omega_n \in \Sigma$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$) and $\varepsilon > 0$,

then given any $A \in \Sigma$ with $\mu(A) < +\infty$ we can find $B \in \mathcal{F}$ such that $\mu(A \triangle B) < \varepsilon$.

Before passing to measurable sets let us mention a few things about the existence of nonmeasurable sets on $\mathbb{R}$.
We have the following result:

PROPOSITION 2.1.46 If we assume the axiom of choice (which is actually part of the axiomatics in this book),

then there exists a set $A \in \mathbb{R}$ which is not Lebesgue measurable (in fact we can find $A \subseteq [0, 1]$ such that $\lambda^*(A) = \lambda^*(I \backslash A) = 1$).

REMARK 2.1.47 A construction of A goes as follows. Let

$$I = \left[\frac{-1}{2}, \frac{1}{2}\right].$$

For $x, y \in I$ write $x \sim y$ if and only if $x - y \in \mathbb{Q}$. We can show that $\sim$ is an equivalence relation.

We can decompose I into the equivalence classes of $\sim$.

Let A be the set containing exactly one member of each equivalence class (here we use the axiom of choice).

Then it can be shown that A is nonmeasurable (for details we refer to Royden (1968), p. 63-64).

Now we turn our attention to measurable functions. We define measurability in the following two contents.

DEFINITION 2.1.48 (a) If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, a function $f : \Omega_1 \to \Omega_2$ is said to be "measurable" (or "(Σ_1, Σ_2)-measurable"),

if for all $A \in \Sigma_2$ $f^{-1}(A) \in \Sigma_1$.

(b) If (Ω, Σ) is a measurable space and Y is a topological space, then $f : \Omega \to Y$ is "measurable", if it is $(\Sigma, \mathcal{B}(Y))$-measurable in the sense of part (a) ($\mathcal{B}(Y)$ is the Borel σ-field of Y).

REMARK 2.1.49 If in case (b) of Definition 2.1.48, $\Omega = X$, X is a topological space and $f\colon X \to Y$, we say that f is "(Borel) measurable", if it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.

If $X = \mathbb{R}^N$, we say that f is "Lebesgue measurable", if it is $(\mathcal{L}_N, \mathcal{B}(Y))$-measurable. Notice that in all cases, when the range space is topological, we use the Borel σ-field.

The reason is that the Lebesgue σ-field on the range space may be too large.

Indeed, there exists a continuous nondecreasing function $f\colon [0,1] \to [0,1]$ and a Lebesgue measurable set $A \subseteq [0,1]$ such that $f^{-1}(A)$ is not Lebesgue measurable (assuming the axiom of choice).

Finally notice that the definition of measurable function depends in no way upon any measure, but only upon a particular σ-field.

PROPOSITION 2.1.50 If X, Y are sets, $f : X \to Y$ and $\mathcal{F}$ is a nonempty family of subsets of Y,

then $\sigma(f^{-1}(\mathcal{F})) = f^{-1}(\sigma(\mathcal{F}))$ (here $f^{-1}(\mathcal{F}) = \{f^{-1}(A) \ : \ A \in \mathcal{F}\}$).

A useful consequence of this proposition 2.1.50 is the following result.

COROLLARY 2.1.51 If (X, Σ_X) and (Y, Σ_Y) are measurable spaces, $f\colon X \to Y$ and $\mathcal{F}$ is a family of subsets of Y such that $\sigma(\mathcal{F}) = \Sigma_Y$,

then f is (Σ_X, Σ_Y)-measurable if and only if $f^{-1}(A \in \Sigma_X$ for all $A \in \mathcal{F}$.

COROLLARY 2.1.52 If X and Y are topological spaces and φ is a homeomorphism of X into Y,

then $\varphi(\mathcal{B}(X)) = \mathcal{B}(\varphi(X))$.

When the range space is $\mathbb{R}$, then we can use Corollary 2.1.51 together with Proposition 2.1.7 to give the following characterization of measurable $\mathbb{R}$-valued functions.

COROLLARY 2.1.53 If (Ω, Σ) is a measurable space and $f : \Omega \to \mathbb{R}$,

then the following conditions are equivalent

(a) f is measurable;

(b) for all $\theta \in \mathbb{R}$, $\{\omega \in \Omega \; : \; f(\omega) \geq \theta\} \in \Sigma$;

(c) for all $\theta \in \mathbb{R}$, $\{\omega \in \Omega \; : \; f(\omega) < \theta\} \in \Sigma$;

(d) for all $\theta \in \mathbb{R}$, $\{\omega \in \Omega \; : \; f(\omega) \leq \theta\} \in \Sigma$.

We shall also need the concept of measurablity for extended real functions. We do this by making the conventions that the singletons $\{+\infty\}$ and $\{-\infty\}$ of the extended real line $\mathbb{R}^*$ are Borel sets.
Then the next proposition becomes clear.

PROPOSITION 2.1.54 If (Ω, Σ) is a measurable space and $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$,

then f is measurable if and only if $f^{-1}(\{+\infty\}), f^{-1}(\{-\infty\}) \in \Sigma$ and $f^{-1}(B) \in \Sigma$ for every $B \in \mathcal{B}(\mathbb{R})$.

PROPOSITION 2.1.55 If (Ω, Σ) is a measurable space, $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ and $D \subseteq \mathbb{R}$ is dense subset, then the following conditions are equivalent

(a) f is measurable;

(b) for all $\theta \in D$, $f^{-1}((\theta, +\infty]) \in \Sigma$;

(c) for all $\theta \in D$, $f^{-1}([\theta, +\infty]) \in \Sigma$;

(d) for all $\theta \in D$, $f^{-1}([-\infty, \theta)) \in \Sigma$;

(e) for all $\theta \in D$, $f^{-1}([-\infty, \theta]) \in \Sigma$.

COROLLARY 2.1.56 If X is a topological space and Σ is a σ-field on X such hat $\mathcal{B}(X) \subseteq \Sigma$,

then all $\mathbb{R}$-valued continuous and all $\mathbb{R}^*$-valued lower (upper) semicontinuous functions defined on X are Σ-measurable.

PROPOSITION 2.1.57 If (Ω, Σ) is a measurable space, $f : \Omega \to \mathbb{R}^*$ is Σ-measurable and $g : \mathbb{R}^* \to \mathbb{R}^*$ is a function such that for all $\theta \in \mathbb{R}$, $g^{-1}([\theta, +\infty]) \cap \mathbb{R}$ is a Borel set,

then $g \circ f : \Omega \to \mathbb{R}^*$ is Σ-measurable.

REMARK 2.1.58 Note that the composition of two Lebesgue measurable functions need not be Lebesgue measurable.

We will outline an example illustrating this. For every $x \in [0,1]$, let $\{a_n\}_{n\geq 1}$ denote its ternary expansion (i.e.

$$x = \sum_{n\geq 1} \frac{a_n}{3^n} = ._3 a_1 a_2 \ldots a_n \ldots$$

with $a_n = 0, 1$ and 2).

Let $n(x)$ be the first index for which $a_n = 1$. If there is no such n, i.e. if $x \in C$, C being the Cantor set (Example 2.1.30), then set $n(x) = +\infty$. Let

$$g(x) = \sum_{k=1}^{n(x)} \frac{a_k}{2^{k+1}} + \frac{1}{2^{n(x)}}.$$

Then g is nondecreasing, continuous and onto $I = [0,1]$. It is called the "Cantor function".

Окончание Замечания 2.1.58

Set

$$f(x) = \frac{1}{2}(g(x) + x).$$

Then f is continuous and strictly increasing from I onto itself. If C is the Cantor ternary set, then $g^{-1}(C)$ is Lebesgue measure and has positive Lebesgue measure. Also there exists $A \subseteq I$ Lebesgue measurable such that $f^{-1}(A)$ is not Lebesgue measurable (so f is the function mentioned in Remark 2.1.49).Let

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M \\ 0, & \text{if } x \notin M \end{cases}$$

and set $h = \chi_A \circ f$. Then χ_A and f are Lebesgue measurable, but h is not. Finally note that the above construction also shows that there exist non-Borelian Lebesgue measurable sets. The set A above is such a set (compare with Example 2.1.33).

The proof of the next proposition is straight forward and we leave it to the reader.

PROPOSITION 2.1.59 (a) If (Ω, Σ) is a measurable space and $f, g \colon \Omega \to \mathbb{R}$ are Σ-measurable functions, then so are the functions

$$\lambda f (\lambda \in \mathbb{R}), f+g, \max\{f,g\}, \min\{f,g\}, fg, |f|.$$

(b) If (Ω, Σ) is a measurable space and $f_n \colon \Omega \to \mathbb{R}^*$, $n \geq 1$, are Σ-measurable, then

$$\liminf_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n, \quad \inf_{n\geq 1} f_n, \quad \sup_{n\geq 1} f_n$$

are all Σ-measurable and $\mathbb{R}^*$-valued functions.

REMARK 2.1.60 Proposition 2.1.59(a) is actually valid also for $\mathbb{R}^*$-valued functions provided we take care that the quantities involved are well defined.

Also from Proposition 2.1.59(b) it follows that the pointwise limit of Σ-measurable $\mathbb{R}^*$-valued functions is Σ-measurable.

In fact we can generalize this as follows.

PROPOSITION 2.1.61 If (Ω, Σ) is a measurable space, (Y, d) is a metric space and $f_n \colon \Omega \to Y$ are Σ-measurable functions such hat for all $\omega \in \Omega, \; f_n(\omega) \to f(\omega)$ in Y,

then f is Σ-measurable too.

REMARK 2.1.62 The result fails if Y is nonmetrizable.

In fact, if $I = [0, 1]$, there is a sequence of continuous functions $f_n \colon I \to I^I$ such that $f_n(x) \to f(x)$ for all $x \in I$ and f is not even Lebesgue measurable.

The following easy fact is very useful in many situations.

PROPOSITION 2.1.63 If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, $\{A_n\}_{n\geq 1} \subseteq \Sigma_1$ are mutually disjoint sets with

$$\bigcup_{n\geq 1} A_n = \Omega_1$$

and $f_n \colon A_n \to \Omega_2$ are (Σ_{A_n}, Σ_2)-measurable (see Definition 2.1.4),

then $f \colon \Omega_1 \to \Omega_2$ defined by $f(\omega) = f_n(\omega)$ if $\omega \in A_n$ is (Σ_1, Σ_2)-measurable.

In contrast to continuous functions, a measurable function f defined on a measurable set $A \subseteq \Omega$ with values in $\mathbb{R}$, can be extended trivially to a measurable function $\widehat{f}$ on Ω by letting $\widehat{f}$ have, for example, some fixed value on $\Omega \backslash A$. What is rather surprising is that this extension is also possible, even if A is not measurable. To do this we need some preparation.

DEFINITION 2.1.64 Let (Ω, Σ) be a measurable space. A function $s \colon \Omega \to \mathbb{R}$ which assumes only a finite number of values $\{r_k\}_{k=1}^n$

is said to be a "simple function", if $A_k = s^{-1}(\{r_k\}) \in \Sigma$ for every $k \in \{1, \dots, n\}$. In this case

$$s = \sum_{k=1}^{n} r_k \chi_{A_k}$$

is called the "standard representation" of s.

The next proposition tells us that on a measurable space the measurable functions are precisely the pointwise limits of sequences of simple functions.

PROPOSITION 2.1.65 If (Ω, Σ) is a measurable space,

then $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is Σ-measurable if and only if there exists a sequence $\{s_n\}_{n\geq 1}$ of simple functions such that $|s_n(\omega)| \leq |f(\omega)|$ for all $\omega \in \Omega$ and all $n \geq 1$ and $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

REMARK 2.1.66 A careful reading of the above proof reveals that if f is bounded,

then $s_n \to f$ uniformly on Ω.

THEOREM 2.1.67 If (Ω, Σ) is a measurable space, $A \subseteq \Omega$ (not necessarily in Σ) and $f\colon A \to \mathbb{R}$ is Σ_A-measurable,

then there exists $\widehat{f}\colon \Omega \to \mathbb{R}$ a Σ-measurable function such that $\widehat{f}|_A = f$.

In fact the result remains valid if $\mathbb{R}$ is replaced by a Polish space Y. We will only state the relevant proposition and for a proof we refer to Dudley (1989), p. 97 or Kuratowski (1966), p. 434. In this context by a simple function we understand a function $s\colon \Omega \to Y$ such that s has finite range and for every $y \in Y$, $s^{-1}(\{y\}) \in \Sigma$.

THEOREM 2.1.68 If (Ω, Σ) is a measurable space, Y is a separable metric space, $A \subseteq \Omega$ nonempty (not necessarily in Σ) and $f\colon A \to Y$ is Σ_A-measurable,

then
(a) there are Σ_A-simple functions $\{s_n\}_{n\geq 1}$ such that $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$;

(b) if Y is complete (i.e. Y is Polish), then f admits a $\widehat{\Sigma}$-measurable extension $\widehat{f}\colon \Omega \to Y$.

Примечание: В п. (b) измеримость продолженной функции следует понимать относительно исходной Σ-алгебры (см. док. теоремы 2.1.67).

In the next theorem, pointwise convergence of measurable functions is shown to be uniform except on small sets.
The result is known "Egorov's theorem"

THEOREM 2.1.69 If $(\Omega, \Sigma\mu)$ is a finite measure space, (Y, d) a metric space, $f_n, f : \Omega \to Y$ are Σ-measurable functions and $f_n(\omega) \to f(\omega)$ μ-a.e. on Ω,

then for any $\varepsilon > 0$ we can find $A \in \Sigma$ with $\mu(A^c) < \varepsilon$ such that $f_n \to f$ uniformly on A, i.e.

$$\lim_{n\to\infty} \sup[d(f_n(\omega), f(\omega)) \ : \ \omega \in A] = 0.$$

DEFINITION 2.1.70 Let $(\Omega, \Sigma\mu)$ be a measure space.

A set $A \in \Sigma$ is called an "atom" of $\mu(A) > 0$ and if $C \in \Sigma, C \subseteq A$, then either $\mu(C) = \mu(A)$ or $\mu(C) = 0$.

If μ has no atoms, then μ is said to be "nonatomic".

EXAMPLE 2.1.71 Let Ω be a set, $\Sigma = 2^{\Omega}$ and $\mu \colon \Sigma \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\mu(A) = \begin{cases} card\ A, & \text{if } A \text{ is finite} \\ +\infty, & \text{if } A \text{ is infinite} \end{cases}$$

Then μ is called "counting measure" and for this measure every singleton is an atom.

The Lebesgue measure λ_N on $\mathbb{R}^N$ $(N \geq 1)$ is nonatomic.

The next theorem shows that a finite positive measure space can have at most a countable family of disjoint atoms.
For a proof we refer to Dunford-Schwartz (1958), p. 308.

THEOREM 2.1.72 If (Ω, Σ) is a finite measure space and $\varepsilon > 0$,

then

$$\Omega = \bigcup_{k=1}^{n} A_k$$

for some $n \in \mathbb{N}$, $\{A_k\}_{k=1}^{n} \subseteq \Sigma$ are mutually disjoint and each A_k is either an atom or $\mu(A_k) \leq \varepsilon$.

The next theorem is known as "Lyapunov's theorem" and has important implications in control theory.
An elementary proof of this result can be found in Halmos (1948), while a more sophisticated proof based on the Krein-Milman theorem (see Theorem 3.5.23) was given by Lindenstrauss (1966).
The infinite dimensional version of this theorem can be found in Theorem 3.10.37.

THEOREM 2.1.73 If (Ω, Σ) is a measurable space and $\mu_k : \Sigma \to \mathbb{R}$, $k = 1, \ldots, n$, are finite nonatomic measures,

then the set

$$R = \{(\mu_k(A))_{k=1}^n \ : \ A \in \Sigma\}$$

is compact and convex in $\mathbb{R}^n$.

An interesting consequence of Lyapunov's theorem is the next result . One interpretation of this theorem is that it is always possible to cut a "nonatomic" cake fairly.

THEOREM 2.1.74 If (Ω, Σ) is a measurable space, $\mu_k : \Omega \to \mathbb{R}$, $k = 1, \dots, n$, are nonatomic probability measures and $\{\theta_m\}_{m=1}^{N} \subseteq \mathbb{R}_+$ with

$$\sum_{m=1}^{N} \theta_m = 1,$$

then there exists a partition $\{A_m\}_{m=1}^{N}$ of Ω such that $\mu_k(A_m) = \theta_m$ for all $k = 1, \dots, n$ and all $m = 1, \dots, N$.