Лекция Т.2.2 — "2.2 Интегрирование и теоремы о сходимости"

Введение в нелинейный функциональный анализ

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In this section we develop a theory of integration based on our studies of measure spaces and measurable functions. First we define the integral of an arbitrary nonnegative measurable function defined on a measure space. The function need not be bounded and the space need not have finite measure. Then we extend the definition to functions that need not be nonnegative.

We start with the definition of an integral of a nonnegative simple function.

DEFINITION 2.2.1 Let (Ω, Σ, μ) be a measure space and $s\colon \Omega \to \mathbb{R}_+$ a simple function,

$$s(\omega) = \sum_{k=1}^{n} r_k \chi_{A_k}(\omega), \quad r_k \geq 0, \quad A_k \in \Sigma, \quad k = 1, \ldots, n.$$

Then we define

$$\int_{\Omega} s(\omega) d\mu(\omega) = \sum_{k=1}^{n} r_k \mu(A_k).$$

If for some $k = 1, \ldots, n$, $r_k = 0$ and $\mu(A_k) = +\infty$, we set $r_k \mu(A_k) = 0$ (according to the usual arithmetic on $\overline{\mathbb{R}}$).

REMARK 2.2.2 This is a well defined notion, namely it is independent of the particular representation of the simple function. So suppose

$$s(\omega) = \sum_{i=1}^{m} a_i \chi_{B_i}(\omega), \quad B_i \in \Sigma \quad i = 1, \ldots, m,$$

is another representation of s. Note that

$$s(\omega) = \sum_{k=1}^{n} \sum_{i=1}^{m} c_{ki} \chi_{A_k \cap B_i},$$

where $c_{ki} = r_k = a_i$. Therefore we have

$$\sum_{k=1}^{n} \sum_{i=1}^{m} c_{ki} \mu(A_k \cap B_i) = \sum_{k=1}^{n} r_k \sum_{i=1}^{m} \mu(A_k \cap B_i) =$$

$$= \sum_{k=1}^{n} r_k \mu(A_k) = \sum_{i=1}^{m} a_i \mu(B_i)$$

(by a symmetric argument). So indeed Definition 2.2.1 is independent of the particular representation of the simple function.

The following proposition is a straightforward consequence of Definition 2.2.1

PROPOSITION 2.2.3 If (Ω, Σ, μ) is a measure space, s and h are nonnegative simple functions on Ω and $c \geq 0$, then

(a) if $s(\omega) = h(\omega)$ μ-a.e. on Ω, then $\int\limits_{\Omega} s d\mu = \int\limits_{\Omega} h d\mu$;

(b) $\int\limits_{\Omega} cs d\mu = c \int\limits_{\Omega} s d\mu$;

(c) $\int\limits_{\Omega} (s+h) d\mu = \int\limits_{\Omega} s d\mu + \int\limits_{\Omega} h d\mu$;

(d) if $s(\omega) \leq h(\omega)$ μ-a.e., then $\int\limits_{\Omega} s d\mu \leq \int\limits_{\Omega} h d\mu$.

Now we can define the integral of a Σ-measurable function $f : \Omega \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$.

DEFINITION 2.2.4 Let (Ω, Σ, μ) be a measure space and $f : \Omega \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ be a Σ-measurable function. The "integral" of f with respect to μ is defined by

$$\int\limits_{\Omega} f(\omega) d\,\mu(\omega) = \sup \left\{ \int\limits_{\Omega} s(\omega) d\mu(\omega) \ : \ s \text{ is simple}, \quad 0 \leq s \leq f \right\}.$$

We say that f is "integrable" if $\int\limits_{\Omega} f d\,\mu < +\infty$.

Finally if $A \in \Sigma$, we can define the integral of f over A by

$$\int\limits_{A} f d\,\mu = \int\limits_{\Omega} f \chi_A d\,\mu.$$

REMARK 2.2.5 According to Definition 2.2.4 the integral of a nonnegative Σ-measurable function always exists; it may be $+\infty$. It is easy to check that properties (a), (b) and (d) of Proposition 2.2.3 still hold. Property (c) also holds, but we need a convergence theorem, which is known as "Monotone Convergence Theorem".

THEOREM 2.2.6 (Monotone Convergence Theorem) If (Ω, Σ, μ) is a measure space and $f_n \colon \Omega \to \overline{\mathbb{R}}_+$ is an increasing sequence of Σ-measurable functions with

$$f = \lim_{n\to\infty} f_n,$$

then

$$\int\limits_\Omega f_n d\mu \uparrow \int\limits_\Omega f d\mu.$$

Combining Proposition 2.1.65 with Theorem 2.2.6, we obtain a useful corollary, which in the context of Banach space valued function is the starting point of Bochner integration (see Section 3.9).

COROLLARY 2.2.7 If (Ω, Σ, μ) is a measure space and $f_n \colon \Omega \longrightarrow \overline{\mathbb{R}}_+$ is a Σ-measurable function, then we can find simple functions $\{s_n\}_{n \geq 1}$ such that $s_n(\omega) \uparrow f(\omega)$ μ-a.e. and

$$\int_\Omega s_n d\mu \uparrow \int_\Omega f d\mu.$$

Now we can prove the additivity property of the integral together with two more useful properties.

PROPOSITION 2.2.8 Let (Ω, Σ, μ) be a measure space.
(a) If $f, g\colon \Omega \to \overline{\mathbb{R}}_+$ are Σ-measurable functions, then

$$\int\limits_{\Omega} (f+g) d\mu = \int\limits_{\Omega} f d\mu + \int\limits_{\Omega} g d\mu;$$

(b)If $f_n\colon \Omega \to \overline{\mathbb{R}}_+$ are Σ-measurable functions, then

$$\int\limits_{\Omega} \left(\sum_{n\geq 1} f_n d\mu \right) = \sum_{n\geq 1} \int\limits_{\Omega} f_n d\mu;$$

(c) If $f\colon \Omega \to \overline{\mathbb{R}}_+$ are Σ-measurable functions, then $m(A) = \int\limits_A f d\mu, \quad A \in \Sigma$ is a measure on Σ.

Now we will complete the definition of the integral by defining the integral of a Σ-measurable, $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$-valued function.

DEFINITION 2.2.9 Let (Ω, Σ) be a measurable space and $f : \Omega \to \mathbb{R}^*$ is a Σ-measurable function. We set $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. These are the "positive" and "negative" parts of the function f and they are both Σ-measurable, nonnegative functions.
Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Using f^+ and f^- we can define the integral of f.

DEFINITION 2.2.10 Let (Ω, Σ, μ) be a measure space and $f : \Omega \to \mathbb{R}^*$ be a Σ-measurable function.

We say that f is "integrable", if both f^+ and f^- are integrable (Definition 2.2.4). In that case we define

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu.$$

We denote the class of μ-integrable functions on Ω by $\mathcal{L}^1(\Omega, \mu)$ or simply by $\mathcal{L}^1(\Omega)$ when no confusion is possible.

REMARKS 2.2.11 Note that $|f| \in \mathcal{L}^1(\Omega)$ whenever $f \in \mathcal{L}^1(\Omega)$. Observe that the quantity

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu.$$

makes sense even if one (but not both) of the quantities

$$\int_\Omega f^+ d\mu, \quad \int_\Omega f^- d\mu.$$

is infinite. Some authors use the term "quasi–integrable" for such functions.

In the next proposition we list some elementary properties of integrable functions. Their proofs are straightforward and are left to the reader.

PROPOSITION 2.2.12 Let (Ω, Σ, μ) is a measure space, $c \in \mathbb{R}$ and $f, g \in \mathcal{L}^1(\Omega)$, then

$$\text{(a)} \quad \left| \int\limits_\Omega f d\mu \leq \int\limits_\Omega |f| d\mu \right|;$$

$$\text{(b)} \int\limits_\Omega (cf + g) d\mu = c \int\limits_\Omega f d\mu + \int\limits_\Omega g d\mu;$$

$$\text{(c) if } f(\omega) \leq g(\omega) \text{ for all } \omega \in \Omega \text{ then } \int\limits_\Omega f d\mu \leq \int\limits_\Omega g d\mu.$$

A statement about $\omega \in \Omega$ will be said to hold "μ-almost everywhere" (μ-a.e. for short) if and only if it holds for all $\omega \notin A$ for some $A \in \Sigma$ with $\mu(A) = 0$.

PROPOSITION 2.2.13 If (Ω, Σ, μ) is a measure space and $f, g\colon \Omega \to \mathbb{R}^*$ are two Σ-measurable functions such that $f(\omega) = g(\omega)$ μ-a.e. on Ω, then

$$\int\limits_{\Omega} f d\mu$$

is defined if and only if

$$\int\limits_{\Omega} g d\mu$$

is defined and

$$\int\limits_{\Omega} f d\mu = \int\limits_{\Omega} g d\mu.$$

REMARK 2.2.14 According to this proposition, μ-null sets play no role in the theory of integration with respect to μ.

Thus in theorems about integrals, even for sequences of functions, the hypotheses need only hold almost everywhere.

So in Proposition 2.2.12(c), it suffices to assume that $f(\omega) \leq g(\omega)$ μ-a.e. on Ω.

PROPOSITION 2.2.15 If (Ω, Σ, μ) is a measure space and $f : \Omega \to \mathbb{R}^*$ is Σ-measurable, then

$$\text{(a) if } f \in \mathcal{L}^1(\Omega), \text{ then } f(\omega) \in \mathbb{R}\ \mu - a.e. \text{ on } \Omega;$$

$$\text{(b) if } f \geq 0\ \mu - a.e. \text{ on } \Omega \text{ and } \int\limits_{\Omega} f d\mu = 0 \text{ then } f = 0\ \mu - a.e. \text{ on } \Omega.$$

We can now state the extended version of the Monotone Convergence Theorem (Theorem 2.2.6).

THEOREM 2.2.16 (Extended Monotone Convergence Theorem)
If (Ω, Σ, μ) is a measure space and $f_n, f : \Omega \to \mathbb{R}^*$ are Σ-measurable functions such that $f_n \uparrow f$ μ-a.e. on Ω, $g \leq f_n$ μ-a.e. on Ω and

$$-\infty < \int\limits_{\Omega} g d\mu$$

(or alternatively $f_n \downarrow f$ μ-a.e.on Ω, $f_n \leq g$ μ-a.e.on Ω and

$$\int\limits_{\Omega} g d\mu < +\infty),$$

then

$$\int\limits_{\Omega} f_n d\mu \uparrow \int\limits_{\Omega} f d\mu \quad \text{(respectively} \quad \int\limits_{\Omega} f_n d\mu \downarrow \int\limits_{\Omega} f d\mu).$$

Using Theorem 2.2.16 we can prove a convergence result that is basic to all of the limit properties of integrals. The result is known as "Fatou's lemma".

THEOREM 2.2.17 (Fatou's Lemma) If (Ω, Σ, μ) is a measure space and $f_n, g : \Omega \to \mathbb{R}^*$ are Σ-measurable functions, then
(a) if $g \le f_n$ μ-a.e. on Ω and $-\infty < \int\limits_\Omega g d\mu$, we have

$$\int\limits_\Omega \liminf_{n\to\infty} f_n d\mu \le \liminf_{n\to\infty} \int\limits_\Omega f_n d\mu;$$

(b) if $f_n \le g$ μ-a.e. on Ω and $\int\limits_\Omega g d\mu < +\infty$, we have

$$\limsup_{n\to\infty} \int\limits_\Omega f_n d\mu \le \int\limits_\Omega \limsup_{n\to\infty} f_n d\mu$$

EXAMPLE 2.2.18 The inequality in Fatou's lemma can be strict. Let $\Omega = \mathbb{R}$, $\Sigma = \mathcal{L}$ (the Lebesgue a σ-field) and $\lambda = \mu$ (the Lebesgue measure). Let $f_n = \chi_{[n,n+1]}$. Then $f_n(\omega) \to 0$ for all $\omega \in \Omega$, but $\int\limits_{\Omega} f_n d\mu = 1$ for all $n \geq 1$. Hence $0 = \int\limits_{\Omega} \lim_{n\to\infty} f_n d\mu < 1 = lim_{n\to\infty} \int\limits_{\Omega} f_n d\mu$.

The next theorem is the main reason that the Lebesgue integral introduced in this section is more powerful than the well-known from calculus Riemann integral . The result is known as the "Lebesgue Dominated Convergence Theorem".

THEOREM 2.2.19 (Lebesgue Dominated Convergence Theorem)
If (Ω, Σ, μ) is a measure space and $f_n \colon \Omega \to \mathbb{R}^*$ is the sequence of Σ-measurable functions, $f_n(\omega) \to f(\omega)$ μ-a.e. on Ω and $f_n(\omega) \leq h(\omega)$ μ-a.e. on Ω with $h \in \mathcal{L}^1(\Omega)$ and

$$\int\limits_{\Omega} f d\mu = \lim_{n\to\infty} \int\limits_{\Omega} f_n d\mu.$$

EXAMPLE 2.2.20 Without the function $h \in \mathcal{L}^1(\Omega)$ controlling the growth of the sequence $\{f_n\}_{n\geq 1}$, Theorem 2.2.19 fails. Indeed, let $\Omega = \mathbb{R}$, $\Sigma = \mathcal{L}$ (the Lebesgue a σ-field) and $\lambda = \mu$ (the Lebesgue measure). Let $f_n = \chi_{[n,2n]}$. Then $f_n(\omega) \to 0$ for all $\omega \in \Omega$, but $\int\limits_{\Omega} f_n d\mu = \int\limits_{\mathbb{R}} \chi_{[n,2n]} d\lambda = n \nrightarrow 0$.

COROLLARY 2.2.21 If (Ω, Σ, μ) is a measure space, $f_n\colon \Omega \to \mathbb{R}^*$ is the sequence of Σ-measurable functions, $f_n(\omega) \to f(\omega)$ μ-a.e. on Ω and $f_n(\omega) \leq h(\omega)$ μ-a.e. on Ω with $h^p \in \mathcal{L}^1(\Omega)$ $(p > 0)$, then $|f|^p \in \mathcal{L}^1(\Omega)$ and

$$\int\limits_{\Omega} |f_n - f|^p d\mu \to 0.$$

Next we will present some generalizations of Fatou's lemma (Theorem 2.2.17) and of the Lebesgue Dominated Convergence Theorem (Theorem 2.2.19). For this purpose we need to introduce a new type of convergence, in general different from the μ-almost everywhere convergence.

DEFINITION 2.2.22 Let (Ω, Σ, μ) be a measure space. A sequence f_n of μ-a.e. $\mathbb{R}$-valued, Σ-measurable functions "convergence in measure" (or in μ-measure if we want to emphasize the dependence on the measure μ) to a Σ-measurable function f if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{\omega \in \Omega \ : \ |f_n(\omega) - f(\omega)| \geq \varepsilon\}) = 0.$$

When μ is a probability measure, then the convergence in μ-measure is called "convergence in probability".

We shall say that the sequence $\{f_n\}_{n\geq 1}$ is "Cauchy (fundamental) in measure" if for every $\varepsilon > 0$

$$\lim_{n,m\to\infty} \mu(\{\omega \in \Omega \ : \ |f_n(\omega) - f_m(\omega)| \geq \varepsilon\}) = 0.$$

REMARK 2.2.23 In Section 5 we will extend this mode of convergence to functions with values in a separable metric space (Definition 2.5.47) . Using Definition 2.2.22 we can readily verify that

(a) $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then $f = g$ μ-a.e.;

(b) if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, then for all $\xi \in \mathbb{R}$, $\xi f_n + g_n \xrightarrow{\mu} \xi f + g$;

(c) if $f_n \xrightarrow{\mu} f$, then $f_n^+ \xrightarrow{\mu} f^+$, $f_n^- \xrightarrow{\mu} f^-$ and $|f_n| \xrightarrow{\mu} |f|$.

EXAMPLE 2.2.24 In general μ-almost everywhere convergence and convergence in μ-measure are distinct notions.

Let $\Omega = \mathbb{R}, \Sigma = \mathcal{L}$ and $\mu = \lambda$ and consider $f_n = \chi_{[n,n+1]}$.
Then $f_n(\omega) \to 0$ for all $\omega \in \Omega$.
On the other hand $\lambda(\{\omega \in \Omega \ : \ f_n(\omega) \geq 1\}) = 1 \nrightarrow 0$.

Also let $\Omega = [0,1]$, $\Sigma = \mathcal{L}([0,1])$ (the Lebesgue σ-field of $[0,1]$) and $\mu = \lambda$ (the Lebesgue measure on $[0,1]$).
Let $g_{rn} = \chi_{\frac{r-1}{n}, \frac{r}{n}}$ for $r = 1, \ldots, n$ and consider the sequence

$$g_{11}, g_{12}, g_{22}, g_{13}, g_{23}, g_{33}, \ldots, g_{1n}, g_{2n}, g_{3n}, \ldots, g_{nn}, \ldots$$

We have that the sequence converges in measure to 0 but it does not converge μ-almost everywhere.

The situation changes if we are in a finite measure space.

Смотрите следующий слайд.

PROPOSITION 2.2.25 If (Ω, Σ, μ) is a finite measure space, then μ-almost everywhere convergence implies convergence in μ-measure.

Although μ-almost everywhere convergence and convergence in μ-measure are in general distinct modes of convergence (Example 2.2.24) , we can always extract from any convergent in measure sequence an almost everywhere convergent subsequence.

PROPOSITION 2.2.26 If (Ω, Σ, μ) is any measure space and $\{f_n\}_{n\geq 1}$ a sequence which convergence in μ-measure, then it has a subsequence which converges μ-almost everywhere to the same limit.

Using Proposition 2.2.26 we can have the first extension of the Lebesgue Dominated Convergence Theorem (Theorem 2.2.19).

THEOREM 2.2.27 (Extended Lebesgue Dominated Convergence Theorem)
If (Ω, Σ, μ) is a measure space,
$f_n\colon \Omega \to \mathbb{R}^*$ is the sequence of Σ-measurable functions,
$f_n \xrightarrow{\mu} f$ and
$|f_n(\omega)| \le h(\omega)$ μ-a.e. on Ω with $h \in \mathcal{L}^1(\Omega)$,

then

$$f \in \mathcal{L}^1(\Omega)$$

and

$$\int\limits_{\Omega} f\, d\mu = \lim_{n\to\infty} \int\limits_{\Omega} f_n\, d\mu.$$

When the measure space is finite, we can extend further Theorem 2.2.27 by using the concept of uniform integrability which is important in probability theory (convergence of martingales) and in integration theory in general.

DEFINITION 2.2.28 Let (Ω, Σ, μ) be a finite measure space and

$$K \subseteq \mathcal{L}^1(\Omega).$$

We say that K is a "uniformly integrable" set

if

$$\lim_{c \to +\infty} \sup_{f \in K} \int_{\{|f| \geq c\}} |f| \, d\mu = 0.$$

REMARK 2.2.29 It follows immediately that if for every $f \in K$ we have $f(\omega) \leq h(\omega)$ μ-a.e. on Ω with $h \in \mathcal{L}^1(\Omega)$, then K is uniformly integrable.

A uniformly integrable set K is $\mathcal{L}^1(\Omega)$-bounded, i.e. we have

$$\sup_K \int_\Omega |f| \, d\mu < \infty.$$

To see this note that given $\varepsilon > 0$, we can find $c > 0$ large enough such that for all $f \in K$ we have

$$\int_\Omega |f| \, d\mu = \int_{\{|f| \geq c\}} |f| \, d\mu + \int_{\{|f| < c\}} |f| \, d\mu \leq \varepsilon + c\mu(\Omega).$$

The next proposition gives an equivalent definition of uniform integrability which we encounter often in textbooks.

PROPOSITION 2.2.30 If (Ω, Σ, μ) is a finite measure space and $K \subseteq \mathcal{L}^1(\Omega)$, then K is uniformly integrable if and only if the following two conditions hold:

(a) K is $\mathcal{L}^1(\Omega)$-bounded, i.e.

$$\sup_K \int_\Omega |f| \, d\mu < \infty;$$

(b) given $\varepsilon > 0$ we can find $\delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$, then

$$\sup_K \int_A |f| \, d\mu < \varepsilon.$$

REMARK 2.2.31 It can be proved that in Proposition 2.2.30, property (a) is a consequence of (b) if the measure μ is nonatomic.

Another characterization of uniformly integrable sets in $\mathcal{L}^1(\Omega)$ is given in the next proposition, known as the "de la Vallée-Poussin Theorem". The most useful part of this result is the implication $(b) \Rightarrow (a)$. For a proof of the proposition we refer to Dellacherie-Meyer (1978) , p. 24-II or Doob (1995), p. 95.

PROPOSITION 2.2.32 If (Ω, Σ, μ) is a finite measure space and $K \subseteq \mathcal{L}^1(\Omega)$, then the following properties are equivalent:

(a) K is uniformly integrable;

(b) there exists a positive function φ defined on $\mathbb{R}_+$ such that

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty \text{ and } \sup_{f \in K} \int_\Omega \varphi(|f|)\, d\mu < \infty.$$

A basic application of uniform integrability is the following extension of Fatou's lemma and of the Lebesgue Dominated Convergence Theorem.

THEOREM 2.2.33 (Extended Fatou's Lemma)
If (Ω, Σ, μ) is a finite measure space and $\{f_n\}_{n\geq 1} \subseteq \mathcal{L}^1(\Omega)$ is a uniformly integrable sequence, then

$$\int\limits_{\Omega} \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int\limits_{\Omega} f_n \, d\mu \leq$$

$$\leq \limsup_{n\to\infty} \int\limits_{\Omega} f_n \, d\mu \leq \int\limits_{\Omega} \limsup_{n\to\infty} f_n \, d\mu.$$

THEOREM 2.2.34 (Extended Lebesgue Dominated Convergence Theorem II)
If (Ω, Σ, μ) is a finite measure space and $\{f_n\}_{n\geq 1} \subseteq \mathcal{L}^1(\Omega)$ is a uniformly integrable sequence such that $f_n \xrightarrow{\mu} f$ as $n \to \infty$, then

$$\int\limits_{\Omega} |f_n - f| \, d\mu \to 0$$

as $n \to \infty$.

In a similar fashion we can also prove the following generalization of Theorem 2.2.34.

THEOREM 2.2.35 If (Ω, Σ, μ) is a finite measure space, $0 < p < \infty$, $f_n \xrightarrow{\mu} f$ and $\{|f_n|^p\}_{n\geq 1}$ is uniformly integrable, then

$$\int_{\Omega} |f_n - f|^p \, d\mu \to 0$$

as $n \to \infty$.

REMARK 2.2.36 If $p = 1$, then we recover Theorem 2.2.34.

Now we generalize the linear space $\mathcal{L}^1(\Omega)$ as follows:

DEFINITION 2.2.37 Let (Ω, Σ, μ) be any measure space and $0 < p < \infty$. By $\mathcal{L}^p(\Omega)$ (or $\mathcal{L}^p(\Omega, \mu)$ we denote the set of all Σ-measurable functions $f : \Omega \to \mathbb{R}^*$ such that

$$\int_\Omega |f|^p \, d\mu < \infty$$

(i.e. $|f|^p \in \mathcal{L}^1(\Omega)$).

If $1 \leq p < \infty$ the quantity

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{1/p}$$

is called the "L^p-norm" or "p-norm" of f (in fact it is a seminorm).

REMARK 2.2.38 If $f, g \in \mathcal{L}^p(\Omega)$ and we set $d_p(f, g) = \|f - g\|_p$, then $d_p(\cdot, \cdot)$ is a semimetric on $\mathcal{L}^p(\Omega)$. We have a metric space as follows.

DEFINITION 2.2.39 Let (Ω, Σ, μ) be any measure space and $0 < p < \infty$. On $\mathcal{L}^p(\Omega)$ we consider the equivalence relation $\sim$ defined by $f \sim g$ if and only if $f = g$ μ-a.e. Let

$$L^p(\Omega) = \mathcal{L}^p(\Omega)/\sim$$

REMARK 2.2.40 On $L^p(\Omega)$ $d_p(\cdot, \cdot)$ is a metric. In fact as it is well known $(L^p(\Omega), d_p)$ is a complete metric space

(i.e. $L^p(\Omega)$ with the "p-norm" $\|\cdot\|_p$ is a Banach space, see Definition 3.1.10(f)).

The next proposition is a byproduct of the proof of the completeness of $(L^p(\Omega), d_p)$ and its proof can be found in Kufner-John-Fučik (1977) , p. 74.

PROPOSITION 2.2.41 If (Ω, Σ, μ) is any measure space, $\{f_n, f\}_{n\geq 1} \subseteq L^p(\Omega)$ $(1 \leq p < \infty)$ and $\|f_n - f\|_p \to 0$, then we can extract a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that $f_{n_k}(\omega) \to f(\omega)$ μ-a.e. on Ω and for all $k \geq 1$ $|f_{n_k}(\omega)| \leq h(\omega)$ μ-a.e. on Ω with $h \in L^p(\Omega)$.

There are two basic inequalities involving the p-norm. These are Holder's inequality and Minkowski's inequality.
Before proving them, we establish a lemma, which can be viewed as a generalization of the classical inequality between the arithmetic and the geometric means.

LEMMA 2.2.42 If x, y are nonnegative real numbers and $0 < \lambda < 1$, then

$$x^\lambda y^{1-\lambda} \leq \lambda x + (1-\lambda)y$$

with equality only if $x = y$.

We can allow also $p = \infty$, provided we define the corresponding norm differently, namely in a pointwise fashion.

DEFINITION 2.2.43 Let (Ω, Σ, μ) be a measure space and $f : \Omega \to \mathbb{R}^*$ be a Σ-measurable function.

The "essential supremum" of f is defined by

$$\textit{ess sup } f = \inf\{c \in \mathbb{R}^* \ : \ \mu(\{\omega \in \Omega \ : \ f(\omega) > c\}) = 0\},$$

i.e. *ess sup* is the smallest number c such that $f(\omega) \le c$ μ-a.e. on Ω.

The space $\mathcal{L}^\infty(\Omega)$ is the collection of all Σ-measurable functions $f : \Omega \to \mathbb{R}^*$ such that *ess sup* $|f| < \infty$.
The quantity $\|f\|_\infty =$ *ess sup* $|f|$ is called the "L^∞ norm" of f .
As before

$$L^\infty(\Omega) = \mathcal{L}^\infty(\Omega)/ \sim$$

(see Definition 2.2.39).

THEOREM 2.2.44 (Hölder Inequality)
If (Ω, Σ, μ) is any measure space, $1 \leq p, q \leq \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1$$

and

$$f \in L^p(\Omega), \quad g \in L^q(\Omega),$$

then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \, d\mu \leq \|f\|_p \|g\|_q.$$

REMARK 2.2.45 When $p = q = 2$, we obtain

$$\int_\Omega |fg| \, d\mu \le \left(\int_\Omega |f|^2 \, d\mu \int_\Omega |g|^2 \, d\mu \right)^{1/2}$$

which is a particular instant of basic inequality in analysis known as the Cauchy-Bunyakovsky–Schwarz inequality
(See Proposition 3.7.4).

The numbers p, q such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

are called ''conjugate exponents''.

THEOREM 2.2.46 (Minkowski Inequality)
If (Ω, Σ, μ) is any measure space and $1 \leq p \leq \infty$ and $f, g \in L^p(\Omega)$,

then

$$f + g \in L^p(\Omega)$$

and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

REMARK 2.2.47 From Theorem 2.2.46 we have that $L^p(\Omega)$ is a vector space with a complete norm $\|\cdot\|_p$, thus a Banach space (Definition 3.1.10 (f)) .
In the above proof we have used the elementary inequality which says that if $a, b \geq 0$ and $p \geq 1$, then

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

If $0 < p < 1$, then $(a+b)^p \leq a^p + b^p$.

An interesting consequence of Hölder's inequality is the following result.

PROPOSITION 2.2.48 If (Ω, Σ, μ) is any measure space, $1 \leq p < r \leq \infty$, then

$$L^r(\Omega) \subseteq L^p(\Omega)$$

Another useful consequence of Hölder's inequality is the next known as "Generalized Hölder's Inequality". We leave the proof to the reader.

PROPOSITION 2.2.49 If (Ω, Σ, μ) is any measure space, $f_k \in L^{p_k}(\Omega)$, $k = 1, \ldots, n$, and

$$\sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{p} \leq 1,$$

then

$$f = f_1 f_2 \cdots f_n \in L^p(\Omega)$$

and

$$\|f\|_p \leq \|f_1\|_{p_1} \|f_1\|_{p_2} \cdots \|f_n\|_{p_n}.$$

A third byproduct of Hölder's inequality is the so-called "Interpolation Inequality" which is very useful in applications.

PROPOSITION 2.2.50 If (Ω, Σ, μ) is any measure space and $f \in L^p(\Omega) \cap L^r(\Omega)$ with $1 \leq p \leq r \leq \infty$,

then

$$f \in L^s(\Omega)$$

for all $p \leq s \leq r$ and

$$\|f\|_s \leq \|f\|_p^{\theta} \|f\|_r^{1-\theta},$$

where

$$0 \leq \theta \leq 1 \qquad \frac{1}{s} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

A third basic inequality associated with integrable functions is the so-called "Jensen's Inequality".

THEOREM 2.2.51 (Jensen's Inequality)
If (Ω, Σ, μ) is a finite measure space, I is an open interval in $\mathbb{R}$, $\varphi \colon I \to \mathbb{R}$ is a convex function, $f \in L^1(\Omega)$ with $f(\Omega) \subseteq I$ and $\varphi \circ f \in L^1(\Omega)$,

then

$$\varphi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f) \, d\mu.$$

So far we have considered integrals with respect to a fixed measure μ. Now we will generalize by allowing the measure to vary too.

DEFINITION 2.2.52 Let (Ω, Σ, μ) be a measurable space and $\{\mu_n, \mu\}_{n\geq 1}$ be set functions on Σ.

We say that μ_n "converge setwise" to μ, if for all $A \in \Sigma$ we have

$$\mu_n(A) \to \mu(A)$$

Using this notion we can have the following generalization of Fatou's lemma.

PROPOSITION 2.2.53 If (Ω, Σ) is a measurable space, $\{\mu_n, \mu\}_{n\geq 1}$ are measures on Σ

such that

$$\mu_n \longrightarrow \mu$$

setwise and $\{f_n\}_{n\geq 1}$ is a sequence of nonnegative Σ-measurable functions such that

$$f_n(\omega) \to f(\omega)$$

for all $\omega \in \Omega$, then

$$\int\limits_{\Omega} f \, d\mu \leq \liminf_{n\to\infty} \int\limits_{\Omega} f_n \, d\mu_n$$

Using this proposition 2.2.53, we can have the following extension of the Lebesgue Dominated Convergence Theorem.

PROPOSITION 2.2.54 If (Ω, Σ) is a measurable space, $\{\mu_n, \mu\}_{n\geq 1}$ are measures on Σ such that $\mu_n \to \mu$ setwise, $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$ are two sequence of Σ-measurable functions satisfying

$$f_n(\omega) \to f(\omega), \quad g_n(\omega) \to g(\omega), \quad |f_n(\omega)| \leq g_n(\omega)$$

for all $\omega \in \Omega$ and

$$\int\limits_\Omega g_n \, d\mu_n \to \int\limits_\Omega g \, d\mu$$

as $n \to \infty$, then

$$\int\limits_\Omega f_n \, d\mu_n \to \int\limits_\Omega f \, d\mu \quad \text{as} \quad n \to \infty.$$

Next we will state two change of variable results for integrable functions. The first is a general result for transformations between measurable spaces and the second is the classical finite dimensional change of variable formula. We start with a definition.

DEFINITION 2.2.55 Let (Ω, Σ_1) and (S, Σ_2) be two measurable spaces, $u\colon \Omega \to S$ be a (Σ_1, Σ_2)-measurable map and μ be a measure on (Ω, Σ_1).

Then

$$\nu(A) = \mu(u^{-1}(A))$$

for all $A \in \Sigma_2$ is a measure on Σ_2, called the "image measure" of μ by u on Σ_2 and it is denoted by μu^{-1}.

Using this notion we can have the first change of variables formula. For a proof we refer to Aliprantis-Border (1994), p. 365.

PROPOSITION 2.2.56 If (Ω, Σ_1), (S, Σ_2) are two measurable space, $u\colon \Omega \to S$ is a (Σ_1, Σ_2)-measurable map, μ is a measure on (Ω, Σ_1) and $\nu = \mu u^{-1}$ is the image measure by u on Σ_2, then

(a) if $f \in L^1(S, \nu)$, we have $f \circ u \in L^1(\Omega, \mu)$ and

$$\int\limits_S f\, d\nu = \int\limits_\Omega (f \circ u)\, d\mu;$$

(b) if ν is σ-finite, $f\colon S \to \mathbb{R}$ is Σ_2-measurable and $f \circ u \in L^1(\Omega, \mu)$, we have

$$f \in L^1(S, \nu)$$

and

$$\int\limits_S f\, d\nu = \int\limits_\Omega (f \circ u)\, d\mu.$$

Recall that if $U \subseteq \mathbb{R}^N$ is open and $u\colon U \to \mathbb{R}^N$ is a function having partial derivatives for every $x \in U$, then the matrix

$$\left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1}^{N}$$

is called the "Jacobian matrix" of u and its determinant is known as the "Jacobian" of u and is denoted by $J_u(x)$.

The next result is the classical finite dimensional change of variables formula. A proof can be found in Parthasarathy (1977), p. 178. For a more general result using Hausdorff measures we refer to Evans-Gariepy (1992), p. 99 and 117.

PROPOSITION 2.2.57 If $A, B \subseteq \mathbb{R}^N$ are Lebesgue measurable sets, $u \colon A \to B$, there exist open sets $U \subseteq A$ and $V \subseteq B$ such hat $u \colon U \to V$ is a diffeomorphism and $\lambda_N(A \setminus U) = \lambda_N(B \setminus V) = 0$, then for every $f \in L^1(B, \lambda_N)$, the function $(f \circ u)|J_u|$ defined a.e. on A, belongs to $L^1(A, \lambda_N)$ and we have

$$\int_B f \, d\lambda_N = \int_A (f \circ u)|J_u| \, d\lambda_N.$$

We conclude this section with a useful criterion for the almost everywhere convergence of a sequence of measurable functions.

PROPOSITION 2.2.58 If (Ω, Σ, μ) is a finite measure space and $f_n, f : \Omega \to \mathbb{R}^*$, $n \geq 1$, are Σ-measurable functions,

then

$$f_n(\omega) \to f(\omega)$$

μ-a.e. on Ω if and only if for every $\delta > 0$

$$\mu\left(\bigcup_{k=n}^{\infty} \{\omega \in \Omega \ : \ |f_k(\omega) - f(\omega)| \geq \delta\}\right) \to 0$$

as $n \to \infty$.