Лекция Т.2.3 — "2.3 Теорема Радона-Никодима"

Введение в нелинейный функциональный анализ

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Thus far we have insisted on set functions with values in $\overline{\mathbb{R}}_+$ (i.e. on measures).
In this section we examine what happens if the set function is allowed to take on both positive and negative values.
The resulting objects are called "signed measures".
We may think of measures as distributions of mass and of signed measures as distributions of electric charge.

DEFINITION 2.3.1 Let (Ω, Σ) be a measurable space. A set function $\mu\colon \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is said to be a "signed measure" if it satisfies the following conditions:

(a) μ takes at most one of the values $+\infty$ and $-\infty$;

(b) $\mu(\emptyset) = 0$;

(c) for any sequence $\{A_n\}_{n\geq 1} \subseteq \Sigma$ of pairwise disjoint sets we have

$$\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$$

(the equality is taken to mean that the series of the right hand side convergence absolutely if $\mu\left(\bigcup_{n\geq 1} A_n\right)$ is finite and it diverges otherwise).

A signed measure with values in $\mathbb{R}$ is said to be "finite".

REMARK 2.3.2 A measure is a special case of signed measure. If μ_1, μ_2 are two measures at least one of which is finite, then $\mu = \mu_1 - \mu_2$ is a signed measure.

Доказательство

Since for any

$$A \in \Sigma, \quad \mu(\Omega) = \mu(A) + \mu(A^c)$$

if there exists a set $A \in \Sigma$ such that $\mu(A) = +\infty$, then $\mu(\Omega) = +\infty$ and if there exists a set $A \in \Sigma$ such that $\mu(A) = -\infty$, then $\mu(\Omega) = -\infty$.

Similarly if for $A \in \Sigma$, $\mu(A) \in \mathbb{R}$ and $B \in \Sigma$, $B \subseteq A$, then $\mu(B) \in \mathbb{R}$.

EXAMPLE 2.3.3 Let (Ω, Σ, μ) be a measure space and $f \in L^1(\Omega, \mu)$. For each $A \in \Sigma$ set

$$m(A) = \int_A f \, d\mu$$

Then from the linearity of the integral and the Lebesgue Dominated Convergence Theorem, we can check that $m(\cdot)$ is a signed measure.
Note that $m = m_1 - m_2$ with m_1, m_2 measures.
Indeed let

$$m_1(A) = \int_A f^+ \, d\mu \quad \text{and} \quad m_2(A) = \int_A f^- \, d\mu$$

for all $A \in \Sigma$.

In the sequel we will see that any signed measure admits such a decomposition.

With a proof similar to that of Proposition 2.1.13, we can have its extension to signed measures.

PROPOSITION 2.3.4 If (Ω, Σ) is a measurable space and $\mu\colon \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is a signed measure, then

(a) if $\{A_n\}_{n\geq 1} \subseteq \Sigma$ is an increasing sequence (i.e. $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$), we have

$$\mu\left(\bigcup_{n\geq 1} A_n\right) = \lim_{n\to\infty} \mu(A_n);$$

(b) if $\{A_n\}_{n\geq 1} \subseteq \Sigma$ is a decreasing sequence (i.e. $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$) such that $\mu(A_n)$ is finite for some $n \geq 1$, we have

$$\mu\left(\bigcap_{n\geq 1} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

DEFINITION 2.3.5 Let (Ω, Σ) be a measurable space and $\mu\colon \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ a signed measure.

A subset A of Ω is a "positive set for μ", if $A \in \Sigma$ and for every $B \subseteq A, B \in \Sigma$, we have $0 \leq \mu(B)$.

Similarly, a subset A of Ω is a "negative set for μ", if $A \in \Sigma$ and for every $B \subseteq A, B \in \Sigma$, we have $\mu(B) \leq 0$.

A subset of Ω which is both positive and negative for μ is said to be a "μ-null set".

REMARK 2.3.6 Every Σ-subset of a positive (resp. negative) set for μ, is again positive (resp. negative) set for μ.

Also a Σ-set is a μ-null set if and only if every Σ-subset of it has μ-measure zero.

Note that a μ-null set has μ-measure zero, but a set of μ-measure zero may well be the union of two sets whose μ-measures are not zero but are negatives of each other.

The role of positive and negative sets for a signed measure μ, is clarified by two basic decomposition theorems:
the "Hahn decomposition theorem" and
the "Jordan decomposition theorem".
We start with a lemma, which is stated in terms of negative sets.
Of course a similar statement also holds for positive sets.

LEMMA 2.3.7 If (Ω, Σ) is a measurable space, $\mu\colon \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ a signed measure and $A \in \Sigma$ such that

$$-\infty < \mu(A) < 0,$$

then there is a negative set B for μ such that

$$B \subseteq A \quad \text{and} \quad \mu(B) \leq \mu(A).$$

Now we are ready for the first decomposition theorem for signed measures.

THEOREM 2.3.8 (Hahn Decomposition Theorem)
If (Ω, Σ) is a measurable space, $\mu\colon \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is a signed measure, then

there exist disjoints sets $A_+, A_- \subseteq \Omega$ such that A_+ is a positive set for μ, A_- is a negative set for μ and

$\Omega = A_+ \cup A_-$.

REMARK 2.3.9 The Hahn decomposition $\{A_+, A_-\}$ is not unique.

For example, let $\Omega = [-1, 1]$, $\Sigma = \mathcal{B}([-1, 1])$ (the Borel σ-field of $[-1, 1]$)) and

$$\mu(A) = \int\limits_A x\lambda(dx)$$

for all $A \in \Sigma$ (λ is the Lebesgue measure on $[-1, 1]$).

Note that $\{[0, 1], [-1, 0)\}$ and $\{(0, 1], [-1, 0]\}$ are both Hahn decompositions of μ.

However, if μ is a signed measure on (Ω, Σ) and $\{A^1_+, A^1_-\}$, $\{A^2_+, A^2_-\}$ are two Hahn decompositions of Ω for μ, then we can easily show that for any $B \in \Sigma$ we have

$$\mu(B \cap A^1_+) = \mu(B \cap A^2_+) \text{ and } \mu(B \cap A^1_-) = \mu(B \cap A^2_-).$$

So in this sense the Hahn decomposition of Ω for μ is unique.

Immediately from Theorem 2.3.8 we obtain the second decomposition theorem for signed measure.

THEOREM 2.3.10 (Jordan Decomposition Theorem)
If (Ω, Σ) is a measurable space and $\mu\colon \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is a signed measure, then

$$\mu = \mu^+ - \mu^-$$

with μ^+, μ^- measures at least one of which is finite.

REMARK 2.3.11 The decomposition of μ given by Theorem 2.3.10 is called "Jordan decomposition" of μ.
The measure μ^+ is called the "positive part" of μ and the measure μ^- is called the "negative part" of μ.

The measure $|\mu| = \mu^+ + \mu^-$ is called the "total variation" of μ.

Note that a set A is positive for μ if $\mu^-(A) = 0$, it is negative for μ if $\mu^+(A) = 0$ and it is μ-null set if $|\mu|(A) = 0$.
For any $A \in \Sigma$ we have

$$-\mu^-(A) \le \mu(A) \le \mu^+(A) \ \text{ and } \ |\mu(A)| \le |\mu|(A).$$

Also it is easy to check that for all $A \in \Sigma$

$$|\mu|(A) = \sup \left\{ \sum_{i=1}^{k} |\mu(A_i)| \ : \ \{A_i\}_{i=1}^{k} \text{ is a } \Sigma\text{-partion of } A, k \in \mathbb{N} \right\}.$$

It is noteworthy that for any $A \in \Sigma$ we have

$$\sup_{B \in \Sigma, B \subseteq A} |\mu(B)| \le |\mu|(A) \le 4 \sup_{B \in \Sigma, B \subseteq A} |\mu(B)|.$$

DEFINITION 2.3.12 Let (Ω, Σ) be a measurable space. By $M(\Sigma)$ we denote the space of all finite signed measures furnished with the total variation norm

$$\|\mu\| = |\mu|(\Omega)$$

for all $\mu \in M(\Sigma)$.

REMARK 2.3.13 By virtue of the inequality at the end of Remark 2.3.11, we see that on $M(\Sigma)$ we can provide another equivalent norm given by

$$\|\mu\|_\infty = \sup\{|\mu(A)| \ : \ A \in \Sigma\}.$$

Note that for any $\mu \in M(\Sigma)$ we have

$$\|\mu\|_\infty = \|\mu\| \leq 4\|\mu\|_\infty$$

PROPOSITION 2.3.14 If (Ω, Σ) is a measurable space, then $(M(\Sigma), \|\cdot\|)$ is a complete normed space (i.e. a Banach space, Definition 3.1.10(f)).

The next result is a remarkable boundedness principle in the space $M(\Sigma)$. For a proof we refer to Dunford-Schwartz (1958), Theorem 8, p. 309.

THEOREM 2.3.15 (Nikodym Boundedness Principle)
If (Ω, Σ) is a measurable space,

$$\{\mu_i\}_{i\in I} \subseteq M(\Sigma)$$

and for each $A \in \Sigma$

$$\sup_{i\in I} |\mu_i(A)| < \infty,$$

then

$$\sup_{i\in I, A\in\Sigma} |\mu_i(A)| < \infty.$$

DEFINITION 2.3.16 Let (Ω, Σ) be a measurable space, μ a measure on Σ and m a signed measure on Σ.

We say that m is "absolutely continuous" with respect to μ, denoted by

$$m \ll \mu,$$

if for every $A \in \Sigma$ for which $\mu(A) = 0$, we also have $m(A) = 0$.

At the other extreme, m is "singular" with respect to μ, denoted

$$m \perp \mu,$$

if there exists a μ-null set $A \in \Sigma$ such that

$$|m|(A^c) = 0.$$

REMARK 2.3.17 From the above Definition 2.3.16 it is clear that if μ is a measure and m_1, m_2 are signed measures on a measurable space (Ω, Σ), than

(a) $m_1 \perp \mu$ and $m_2 \perp \mu$ imply $(m_1 + m_2) \perp \mu$;

(b) $m_1 \ll \mu$ and $m_2 \perp \mu$ imply $m_1 \perp m_2$;

(c) $m_1 \ll \mu$ and $m_1 \perp \mu$ imply $m_1 \equiv 0$;

(d) $m_1 \ll \mu$ if and only if $|m_1| \ll \mu$.

PROPOSITION 2.3.18 If (Ω, Σ) is a measurable space, μ is a measure on Σ and m is a finite signed measure on Σ, then

$$m \ll \mu$$

if and only if

$$\lim_{\mu(A) \to 0} m(A) = 0.$$

Before passing to the main theorem of this section, which is the celebrated Radon-Nikodym Theorem, we want to establish two useful convergence theorems for sequences in $M(\Sigma)$.

The approach that we will follow, is due to Saks. Let (Ω, Σ, μ) be a measure space. On Σ we define the semimetric

$$d_\mu(A, B) = \mu(A \triangle B)$$

for every $A, B \in \Sigma$. According to Remark 1.4.2 if we introduce the equivalence relation $\sim$ on Σ, defined by

$$A \sim B$$

if and only if

$$\mu(A \triangle B) = 0,$$

then on

$$\Sigma(\mu) = \Sigma/\sim,$$

d_μ is a metric.

Note that for all $A, B \in \Sigma(\mu)$ we have

$$d_\mu(A, B) = \|\chi_A - \chi_B\|_{L^1(\Omega,\mu)}$$

With this observation the proof of the next proposition becomes straight forward and so it is left to the reader

PROPOSITION 2.3.19 If (Ω, Σ, μ) is a measure space and d_μ, $\Sigma(\mu)$ are defined as above, then

$$(\Sigma(\mu), d_\mu)$$

is a complete metric space and the set theoretic operations

$$(A, B) \to A \cup B, \quad (A, B) \to A \cap B, \quad (A, B) \to A \triangle B$$

are continuous from

$$\Sigma(\mu) \times \Sigma(\mu)$$

into $\Sigma(\mu)$ and so is the complementation operation

$$A \to A^c$$

from $\Sigma(\mu)$ to itself.

Also using Proposition 2.3.18 we have the following result

PROPOSITION 2.3.20 If (Ω, Σ, μ) is a measure space, d_μ and $\Sigma(\mu)$ are defined as above and $m \in M(\Sigma)$, then

$$m \ll \mu$$

if and only if

$$m\colon \Sigma(\mu) \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$$

is continuous.

Now we will study convergence of sequences in $M(\Sigma)$ by means of viewing them as continuous functions on the complete metric space $(\Sigma(\mu), d_\mu)$. The next theorem is one of the most important results in this direction.

THEOREM 2.3.21 (Vitali-Hahn-Saks Theorem)
If (Ω, Σ, μ) is a finite measure apace,

$$\{m_n\}_{n\geq 1} \subseteq M(\Sigma),$$

for each $n \geq 1$,

$$m_n \ll \mu$$

and for every $A \in \Sigma$

$$\lim_{n\to\infty} m_n(A)$$

exists, then

$$m(A) = \lim_{n\to\infty} m_n(A), \quad A \in \Sigma,$$

is a signed measure in $M(\Sigma)$ and $m \ll \mu$.

Another result in this direction is the following theorem, which actually partially extends Theorem 2.3.21.

THEOREM 2.3.22 (Nikodym's Convergence Theorem)
If (Ω, Σ) is a measurable space, $\{m_n\}_{n\geq 1} \subseteq M(\Sigma)$, and for every $A \in \Sigma$,

$$\lim_{n\to\infty} m_n(A)$$

exists, then

$$m(A) = \lim_{n\to\infty} m_n(A), \quad A \in \Sigma,$$

m is an element of $M(\Sigma)$ and $\{m_n\}_{n\geq 1}$ is uniformly σ-additive

(i.e. if $\{A_k\}_{k\geq 1} \subseteq \Sigma$ is decreasing and $\cap_{k\geq 1} A_k = \emptyset$, then

$$\lim_{k\to\infty} \sup_{n\geq 1} m_n(A_k) = 0).$$

We know that if (Ω, Σ, μ) is a measure space and $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is Σ-measurable such that f^+ or f^- is integrable, then

$$m(A) = \int_A f(\omega) d\mu(\omega), \ A \in \Sigma,$$

is a signed measure.
The Radon-Nikodym theorem establishes the converse.
Namely given a signed measure under what conditions we can write it as the indefinite integral of a Σ-measurable function.
The Radon-Nikodym theorem is one of the most important results in analysis and large areas of it are based on this theorem.

THEOREM 2.3.23 (Radon-Nikodym Theorem)
If (Ω, Σ, μ) is a σ-finite measure space,
m is a finite signed measure on Σ and $m \ll \mu$,

then

there exists a μ-integrable function $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ such that

$$m(A) = \int\limits_A f(\omega) d\mu(\omega) \quad \text{for all} \quad A \in \Sigma.$$

If g is another such function, then $f(\omega) = g(\omega)$ μ-a.e. on Ω.

COROLLARY 2.3.24 If the hypotheses of Theorem 2.3.23 hold and m is a measure, then $f(\omega) \geq 0$ μ-a.e. on Ω.

REMARK 2.3.25 The function $f \in L^1(\Omega, \mu)$ obtained in Theorem 2.3.23 is called the "Radon-Nikodym derivative" or "density" of m with respect to μ and is written as

$$\frac{dm}{d\mu}.$$

Also from the proof it is clear that the assumption on m can be weakened and assume that $|m|$ is σ-finite.

In this case f is only Σ-measurable, μ-a.e. finite valued but not μ-integrable.

If μ is a σ-finite measure and m a finite signed measure on the σ-field Σ, then m may be neither absolutely continuous nor singular with respect to μ. Nevertheless the two concepts of absolute continuity and singularity suffice to describe the relation between μ and m in the sense that m can be written as the sum of two signed measures, one which is absolutely continuous with respect to μ and the other which is singular with respect to μ.

This is the third decomposition theorem for signed measures that we will prove.

THEOREM 2.3.26 (Lebegue Decomposition Theorem)
If (Ω, Σ, μ) is a σ-finite measure space and m is a signed measure on Σ such that $|m|$ is σ-finite, then

$$m = m_1 + m_2,$$

with $m_1,\ m_2$ signed measures on Σ such that

$$m_1 \ll \mu, \qquad m_2 \perp \mu.$$

REMARK 2.3.27 The decomposition of m established in Theorem 2.3.26 is known as "Lebesgue decomposition".

For a finite measure m on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we can go a step further and obtain a decomposition

$$m = m_1 + m_2 + m_3,$$

where m_1 is a discrete measure on $\mathcal{B}(\mathbb{R})$, m_2 a continuous measure which is singular with respect to the Lebesgue measure λ on $\mathbb{R}$ and m_3 is absolutely continuous with respect to λ.

A finite or σ-finite measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is continuous if

$$\nu(\{x\}) = 0$$

for all $x \in \mathbb{R}$.
It is discrete if there is a countable set $D \subseteq \mathbb{R}$ such that $\nu(D^c) = 0$.

Продолжение Замечания 2.3.27

In fact if

$$C = \{x \in \mathbb{R} \ : \ m(\{x\}) \neq 0 \,],$$

then from Theorem 2.1.72 we know that C is countable. Then m_1 in the above decomposition is defined by

$$m_1(A) = m(A \cap C)$$

for all $C \in \mathcal{B}(\mathbb{R})$), while m_2 and m_3 are the singular and absolutely continuous (with respect to the Lebesgue measure λ) parts of the measure

$$A \to m(A \cap C^c).$$

It is easy to check that this decomposition is unique too.

In our discussion of the Radon-Nikodym theorem, we have insisted that μ is at most a σ-finite measure. The reason for this is the following.

EXAMPLE 2.3.28 The Radon-Nikodym theorem fails if μ is not σ-finite. Let Ω be an uncountable set and let Σ be the σ-field consisting of all countable subsets of Ω and their complements. Let μ be the "counting measure" on σ, i.e.

$$\mu(A) = \begin{cases} n, & \text{if } A \text{ is a finite set with } n \text{ elements} \\ +\infty, & \text{if } A \text{ is an infinite set.} \end{cases}$$

Evidently μ is not σ-finite. Also let $m\colon \Sigma \to \mathbb{R}_+$ be defined by

$$m(A) = \begin{cases} 0, & \text{if } A \text{ is countable} \\ 1, & \text{otherwise.} \end{cases}$$

Then m is a measure and $m \ll \mu$. However, as it is easy to check, there is no Radon-Nikodym derivative of m with respect to μ.

Next we will prove some facts about the relationship between a finite signed measure and its total variation.

PROPOSITION 2.3.29 If (Ω, Σ, μ) is a measure space, $f \in L^1(\Omega, \mu)$ and

$$m(A) = \int_A f \, d\mu(\omega)$$

for all $A \in \Sigma$,

then

$$|m|(A) = \int_A |f| \, d\mu$$

for all $A \in \Sigma$.

COROLLARY 2.3.30 If (Ω, Σ) is a measurable space and ν is a finite signed measure on Σ, then

$$\left|\frac{d\nu}{d|\nu|}\right| = 1$$

$|\nu|$-a.e. on Ω.

Earlier, starting from a measure space (Ω, Σ, μ) we defined a metric on $\Sigma(\mu)$ (a semimetric on Σ) by setting

$$d_\mu(A, B) = \mu(A \triangle B)$$

for all $A, B \in \Sigma$.
Let us conclude this section with a useful application of this metric.

PROPOSITION 2.3.31 $(\Sigma(\mu), d_\mu)$ is a separable metric space if and only if $L^1(\Omega, \mu)$ is separable.