Лекция Т.2.4 — "2.4 Произведение мер"

Введение в нелинейный функциональный анализ

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The Lebesgue measure on $\mathbb{R}^N$ is in a sense the product of N-copies of the one-dimensional Lebesgue measure, since the volume of an N-dimensional rectangle is the product of the lengths of the sides. In this section we extend this idea to a general setting.

DEFINITION 2.4.1 Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be two measurable spaces. For $A \in \Sigma_1$ and $B \in \Sigma_2$, the set $A \times B$ is said to be a "measurable rectangle".

The smallest σ-field of subsets of $\Omega_1 \times \Omega_2$ which contains all measurable rectangles (i.e. $\sigma(\mathcal{R})$ where $\mathcal{R}$ is the collection of all measurable rectangles) is denoted by $\Sigma_1 \times \Sigma_2$ and it is called the "product σ-field".

Given $C \subseteq \Omega_1 \times \Omega_2$ and $\omega_1 \in \Omega_1$, $\omega_{\in}\Omega_2$, we set

$$C(\omega_1) = \{\omega_2 \in \Omega_2 \ : \ (\omega_1, \omega_2) \in C\}$$

and

$$C(\omega_2) = \{\omega_1 \in \Omega_1 \ : \ (\omega_1, \omega_2) \in C\}$$

and these sets are called the "Ω_1-section" and the "Ω_2-section" of C, respectively.

Similarly for a function f on $\Omega_1 \times \Omega_2$ and $\omega_1 \in \Omega_1$, $\omega_{\in}\Omega_2$, we put

$$f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2) \ \text{ and } \quad f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$$

and these functions of only one variable are called the "Ω_1-section" and the "Ω_2-section" of f, respectively.

REMARK 2.4.2 Note that $\Sigma_1 \times \Sigma_2$ is not the Cartesian product of the σ-field, although the notation may suggest so.

If

$$C,\ C_i \subseteq \Omega_1 \times \Omega_2, \quad i \in I$$

and $\omega_1 \in \Omega_1$, then

$$\bigcup_{i \in I} C_i(\omega_1) = \left(\bigcup_{i \in I} C_i \right)(\omega_1),$$

$$\bigcap_{i \in I} C_i(\omega_1) = \left(\bigcap_{i \in I} C_i \right)(\omega_1),$$

$$(C^c)(\omega_1) = C(\omega_1)^c$$

and

$$(\chi_C)(\omega_1) = \chi_{C(\omega_1)}.$$

PROPOSITION 2.4.3 If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, $C \in \Sigma_1 \times \Sigma_2$ and $f : \Omega_1 \times \Omega_2 \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ is $\Sigma_1 \times \Sigma_2$-measurable function, then

(a) for every $\omega_1 \in \Omega_1$ and every $\omega_2 \in \Omega_2$ we have

$$C(\omega_1) \in \Sigma_2 \qquad C(\omega_2) \in \Sigma_1;$$

(b) for every $\omega_1 \in \Omega_1$, f_{ω_1} is Σ_2-measurable
and for every $\omega_2 \in \Omega_2$, f_{ω_2}, is Σ_1-measurable.

Before stating and proving the first theorem of this section, we state a lemma whose proof is straightforward and it is left to the reader.

LEMMA 2.4.4 If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, then

(a) the family $\mathcal{F}$ of all finite pairwise disjoint unions of measurable rectangles is a field of subsets of $\Omega_1 \times \Omega_2$;

(b) $\Sigma_1 \times \Sigma_2$ is the smallest monotone class containing $\mathcal{F}$.

Примечание.
Families that satisfy (a) and (b) of Proposition 2.1.9 are called "monotone classes":

(a) if $\{A_n\}_{n\geq 1} \subseteq \mathcal{S}$ is increasing, then $\bigcup_{n\geq 1} A_n \in \mathcal{S}$; or

(b) if $\{A_n\}_{n\geq 1} \subseteq \mathcal{S}$ is decreasing, then $\bigcap_{n\geq 1} A_n \in \mathcal{S}$.

THEOREM 2.4.5 If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces and $C \in \Sigma_1 \times \Sigma_2$, then

(a) $\omega_1 \to \mu_2(C(\omega_1))$ is Σ_1-measurable;

(b) $\omega_2 \to \mu_1(C(\omega_2))$ is Σ_2-measurable;

(c)

$$\int\limits_{\Omega_1} \mu_2(C(\omega_1))\, d\mu_1 = \int\limits_{\Omega_2} \mu_1(C(\omega_2))\, d\mu_2.$$

This theorem leads to the definition of the product measure which is made through the next result.

THEOREM 2.4.6 If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces and

$$\mu_1 \times \mu_2 \colon \Sigma_1 \times \Sigma_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

is defined by

$$(\mu_1 \times \mu_2)(C) = \int\limits_{\Omega_1} \mu_2(C(\omega_1))\, d\mu_1 = \int\limits_{\Omega_2} \mu_1(C(\omega_2))\, d\mu_2.$$

(see Theorem 2.4.5(c)),

then

$\mu_1 \times \mu_2$ is a σ-finite measure and for every measurable rectangle $A \times B$, we have

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\mu_2(B)$$

(recall that $0 \cdot \infty = 0$).

DEFINITION 2.4.7 The σ-finite measure

$$\mu_1 \times \mu_2 \colon \Sigma_1 \times \Sigma_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

established in Theorem 2.4.6 is called the "product" of μ_1 and μ_2.

For every measurable rectangle $A \times B$ we have

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\mu_2(B).$$

REMARK 2.4.8 The product measure $\mu_1 \times \mu_2$ is uniquely determined by the requirements that it is a measure on $\Sigma_1 \times \Sigma_2$ and that

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\mu_2(B).$$

for every measurable rectangle $A \times B$.

Now we are ready to examine the relations between integrals on a product space and integrals on the component spaces.
The basic result in this direction is the ''Fubini-Tonelli theorem'', which enables us to evaluate integrals with respect to product measures in terms of iterated integrals.

PROPOSITION 2.4.9 If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces and

$$f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

is $\Sigma_1 \times \Sigma_2$-measurable function, then

(a)

$$\omega_2 \to \int\limits_{\Omega_1} f(\omega_1, \omega_2)\, d\mu_1$$

is Σ_2-measurable and

$$\omega_1 \to \int\limits_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2$$

is Σ_1-measurable;

(b)

$$\int\limits_{\Omega_1 \times \Omega_2} f\, d(\mu_1 \times \mu_2) = \int\limits_{\Omega_2} \left(\int\limits_{\Omega_1} f_{\omega_2}(\omega_1)\, d\mu_1 \right) d\mu_2 = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f_{\omega_1}(\omega_2)\, d\mu_2 \right) d\mu_1$$

THEOREM 2.4.10 (Fubini-Tonelli Theorem)
If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces,

$$f \in \mathcal{L}(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$$

and

$$h_1(\omega_1) = \int\limits_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2, \qquad h_2(\omega_2) = \int\limits_{\Omega_1} f(\omega_1, \omega_2)\, d\mu_1,$$

then

$$h_1 \in \mathcal{L}(\Omega_1, \mu_1), \qquad h_2 \in \mathcal{L}(\Omega_2, \mu_2)$$

and

$$\int\limits_{\Omega_1 \times \Omega_2} f\, d(\mu_1 \times \mu_2) = \int\limits_{\Omega_1} h_1\, d\mu_1 = \int\limits_{\Omega_2} h_2\, d\mu_2.$$

REMARK 2.4.11 The measure space $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$ is seldom complete, even if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are both complete measure spaces.

Indeed, if complete $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ are two σ-finite measure spaces such that there exist

$$A \subseteq \Omega_1 \quad A \notin \Sigma_1$$

and a nonempty set $B \in \Sigma_2$ with $\mu_2(B) = 0$, then the σ-finite measure space

$$(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$$

is incomplete.

In particular, the σ-finite measure space

$$(\mathbb{R}^2, \mathcal{L} \times \mathcal{L}, \lambda_2 = \lambda \times \lambda)$$

is incomplete.

Let $(\Omega_1 \times \Omega_2, \overline{\Sigma_1 \times \Sigma_2}, \overline{\mu_1 \times \mu_2})$ denote the completion of the measure space $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_1)$ (see Proposition 2.1.36).
The next lemma allows us to extend the Fubini-Tonelli theorem to this completed product measure space.

LEMMA 2.4.12 If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are σ-finite measure spaces, $C \in \Sigma_1 \times \Sigma_2$ with

$$(\mu_1 \times \mu_2)(C) = 0$$

and $D \subseteq C$,

then

$$\mu_1(D(\omega_2)) = 0$$

μ_2-a.e. on Ω_2 and

$$\mu_2(D(\omega_1)) = 0$$

μ_1-a.e. on Ω_1

THEOREM 2.4.13 (Extended Fubini-Tonelli Theorem)
If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are complete σ-finite measure spaces and $f\colon \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is a $\overline{\Sigma_1 \times \Sigma_2}$-measurable function, then
(a) for μ_2-almost all $\omega_2 \in \Omega_2$ the function

$$\omega_1 \mapsto f(\omega_1, \omega_2)$$

is Σ_1-measurable and for μ_1-almost all $\omega_1 \in \Omega_1$ the function

$$\omega_2 \mapsto f(\omega_1, \omega_2)$$

is Σ_2-measurable;
(b) the function

$$\omega_2 \mapsto \int\limits_{\Omega_1} f(\omega_1, \omega_2)\, d\mu_1$$

is Σ_2-measurable and the function

$$\omega_1 \mapsto \int\limits_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2$$

is Σ_1-measurable; (см. продолжение на следующем слайде:)

Теорема 2.4.13

Условие (c)

$$(c) \int\limits_{\Omega_1\times\Omega_2} f\, d(\overline{\mu_1 \times \mu_2}) = \int\limits_{\Omega_2} \left(\int\limits_{\Omega_1} f_{\omega_2}(\omega_1)\, d\mu_1 \right) d\mu_2 =$$

$$\int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f_{\omega_1}(\omega_2)\, d\mu_2 \right) d\mu_1.$$

Next we will present a generalized version of the Fubini-Tonelli theorem using transition measures.

Transition measures (in particular transition probabilities) are the main tool in the relaxation of control systems (Section A.4.1) and in stochastic games (Section A.5.5).

DEFINITION 2.4.14 Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be two measurable spaces. A map $m\colon \Omega_1 \times \Sigma_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is said to be a "transition measure" if

(a) for every $B \in \Sigma_2$, the function $\omega_1 \mapsto m(\omega_1, B)$ is Σ_1-measurable;

(b) for every $\omega_1 \in \Omega_1$, the set function $B \mapsto m(\omega_1, B)$ is σ-finite measure on Σ_2.

We say that m is a "σ-finite transition measure" if

$$\Omega_2 = \bigcup_{n\geq 1} A_n, \quad A_n \in \Sigma_2$$

and for all $\omega_1 \in \Omega_1$ we have

$$m(\omega_1, A_n) < +\infty$$

for all $n \geq 1$.
Finally we say that m is a "transition probability", if for all $\omega_1 \in \Omega_1$ we have

$$m(\omega_1, \Omega_2) = 1.$$

REMARK 2.4.15 Transition probabilities have the following interpretation: there are two systems whose states are described by the points in the sets Ω_1 and Ω_2.

The statistical behavior of the outcomes of the second system depends on the state of the first system.

If the state of the first system is $\omega_1 \in \Omega_1$, then the probability that the state of the second system is in a set B is described by the transition probability $m(\omega_1, B)$.

EXAMPLES 2.4.16 (a) Let μ be a σ-finite measure on (Ω_2, Σ_2) and set

$$m(\omega_1, B) = \mu(B)$$

for all $\omega_1 \in \Omega_1$ and all $B \in \Sigma_2$. Then m is trivially a transition measure.

(b) Let $f\colon [0,1] \times [0,1] \to \mathbb{R}_+$ be a continuous function and let

$$m(x, B) = \int_B f(x, y)\, d\lambda(y),$$

where λ denotes the Lebesgue measure on $[0,1]$. Then m is a transition measure.

Продолжение Примера 2.4.16

(c) $P = (p_{ij})_{i,j=1}^n$ is an $n \times n$ matrix with nonnegative entries (i.e. $p_{ij} \geq 0$ for all $1 \leq i, j \leq n$) and

$$\sum_{j=1}^n p_{ij} = 1$$

(i.e. each row adds up to 1). Let

$$\Omega_1 = \Omega_2 = \{1, 2, \ldots, n\}$$

and for any $k \in \Omega_1$ and $B \subseteq \Omega_2$ set

$$m(k, B) = \sum_{j \in B} p_{ij}.$$

Then m is a transition probability and the matrix P is called the associated transition probability matrix.

PROPOSITION 2.4.17 If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces,

$$f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

is a $\Sigma_1 \times \Sigma_2$-measurable function and $m(\cdot, \cdot)$ is a σ-finite transition measure on $\Omega_1 \times \Sigma_2$, then

$$\omega_1 \mapsto \int\limits_{\Omega_2} f(\omega_1, \omega_2) m(\omega_1, d\omega_2)$$

is a Σ_1-measurable function from Ω_1 to $\overline{\mathbb{R}}_+$.

DEFINITION 2.4.18 Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be measurable spaces. A σ-finite transition measure on (Ω_1, Σ_2) m is said to be "uniformly σ-finite", if there exist sequences

$$\{B_k\}_{k\geq 1} \subseteq \Sigma_1$$

and

$$\{D_n\}_{n\geq 1} \subseteq \Sigma_2$$

such that

$$\bigcup_{k\geq 1} B_k = \Omega_1, \quad \bigcup_{n\geq 1} D_n = \Omega_2$$

and

$$\sup_{\omega_1 \in B_k} m(\omega_1, D_n) < +\infty$$

for all $k, n \geq 1$.

PROPOSITION 2.4.19 If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, m is a uniformly σ-finite transition measure on (Ω_1, Σ_2), μ is σ-finite measure on (Ω_1, Σ_1) and $C \in \Sigma_1 \times \Sigma_2$, then

$$\nu(C) = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} \chi_C(\omega_1, \omega_2) m(\omega_1, d\omega_2) \right) \, d\mu$$

is well defined and it is a σ-finite measure on $\Sigma_1 \times \Sigma_2$.
Moreover, if

$$f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$$

is a $\Sigma_1 \times \Sigma_2$-measurable, then

$$\int\limits_{\Omega_1 \times \Omega_2} f \, d\nu = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f(\omega_1, \omega_2) m(\omega_1, d\omega_2) \right) \, d\mu.$$

REMARK 2.4.20 If $m(\omega_1, B) = m(B)$ (i.e. is independent of ω_1, with m σ-finite measure on Σ_2, then the σ-finite measure ν established in Proposition 2.4.19 has the property that

$$\nu(A \times B) = \mu(A)m(B)$$

for all $A \in \Sigma_1$ and $B \in \Sigma_2$,
i.e. ν is the product of μ and m
(see Definition 8.4.7).

Now we can state the generalized version of the Fubini-Tonelli theorem.

THEOREM 2.4.21 (Generalized Fubini-Tonelli Theorem)
If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, μ is a σ-finite measure on Σ_1, m is a uniformly σ-finite transition measure on $\Omega_1 \times \Sigma_2$, ν is the σ-finite measure on $\Sigma_1 \times \Sigma_2$ obtained in Proposition 2.4.19 and $f \in L^1(\Omega_1 \times \Omega_2, \nu)$, then

$$\int\limits_{\Omega_2} |f(\omega_1, \omega_2)| m(\omega_1, d\omega_2) < +\infty$$

μ-a.e. on Ω_1;

$$\int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} |f(\omega_1, \omega_2)| m(\omega_1, d\omega_2) \right) d\mu < +\infty$$

and

$$\int\limits_{\Omega_1 \times \Omega_2} f \, d\nu = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f(\omega_1, \omega_2) m(\omega_1, d\omega_2) \right) \, d\mu.$$

REMARK 2.4.22 If $m(\omega_1, B) = m(B)$ (i.e. m is independent of $\omega_1 \in \Omega_1$) and m is σ-finite measure on Σ_2, then Theorem 2.4.21 reduces to the classical Fubini-Tonelli theorem (Theorem 2.4.10) .

The foregoing definitions and properties readily extend to any finite number of sets and measurable spaces.
In the infinite case some of the definitions have to be modified in order to preserve these properties (compare with the corresponding topological situation, see Section 1.2).

DEFINITION 2.4.23 Let $\{(\Omega_n, \Sigma_n)\}_{n\geq 1}$ be measurable spaces. Set

$$\Omega = \prod_{n\geq 1} \Omega_n.$$

If

$$B_k \subseteq \prod_{n=1}^{k} \Omega_n,$$

then the set

$$B_k \times \prod_{n\geq k+1} \Omega_n$$

is called a "cylinder" with base B_k.
The cylinder is said to be "measurable" if

$$B_k \in \prod_{n=1}^{k} \Sigma_n.$$

Продолжение Определения 2.4.23

If

$$B_k = \prod_{n=1}^{k} A_n$$

with $A_n \in \Sigma_n$ for $n = 1, 2, \ldots, k$, then B_k is said to be a "product measurable cylinder" or "interval" in Ω with sides

$$\{A_n\}_{n=1}^{k}.$$

The minimal σ-field over the measurable cylinders is called the "product" of the σ-fields $\{\Sigma_n\}_{n\geq 1}$ and it is denoted by

$$\prod_{n\geq 1} \Sigma_n.$$

REMARK 2.4.24

$$\prod_{n\geq 1} \Sigma_n.$$

is also the minimal σ-field over the product measurable cylinders.
If all Σ_n coincide with a fixed σ-field Σ, then

$$\prod_{n\geq 1} \Sigma_n.$$

is denoted by

$$\Sigma^\infty.$$

Also if all Ω_n coincide with a fixed set Ω, then

$$\prod_{n\geq 1} \Omega_n.$$

is denoted by

$$\Omega^\infty.$$

In this setting the classical product measure theorem (Theorem 2.4.6) extends as follows (for a proof we refer to Dudley (1989), p. 201).

THEOREM 2.4.25 If $\{(\Omega_n, \Sigma_n, \mu_n)\}_{n\geq 1}$ are probability spaces,

$$\Omega = \prod_{n\geq 1} \Omega_n \quad \text{and} \quad \Sigma = \prod_{n\geq 1} \Sigma_n,$$

then there exists a unique probability measure μ on Σ such that

$$\mu(\{\omega = (\omega_k)_{k\geq 1} \in \Omega \ : \ \omega_1 \in A_1, \ldots, \omega_n \in A_n\}) = \prod_{k=1}^{n} \mu_k(A_k)$$

for all $n \geq 1$ and all $A_k \in \Sigma_k$, $k = 1, 2, \ldots$.
We call μ the product of the μ_n and write

$$\mu = \prod_{n\geq 1} \mu_n.$$

Thus far our approach has been to introduce integration using the concept of measure.
Now we will reverse this process and given an "integral" operation with some suitable properties we will show that it can be represented as the integral with respect to some measure.
This integral is known as the "Daniell Integral".
So let V be a family of $\mathbb{R}$-valued functions on some set Ω.
We assume that V is a vector lattice, i.e. a vector space (i.e. if $f, g \in V$ and $c \in \mathbb{R}$, then $cf + g \in V$) and if $f, g \in V$, then $f \vee g = \max\{f, g\} \in V$.
Note that this implies that we have also that for any $f, g \in V$, $f \wedge g = \min\{f, g\} \in V$.
Indeed observe that

$$f \wedge g = -((-f) \vee (-g)).$$

EXAMPLE 2.4.26 For any measure space (Ω, Σ, μ), $L^p(\Omega)$, $(1 \leq p \leq \infty)$ is a vector lattice.

Also if X is a topological space, the space $C_b(X)$ of all bounded, continuous $\mathbb{R}$-valued functions on X is a vector lattice.

However, $C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \ : \ f$ is continuously differentiable$\}$ is a vector space but not a vector lattice.

DEFINITION 2.4.27 Given a set Ω and a vector lattice V of real functions on Ω, a "Daniell functional" or "Daniell integral", is a function $I\colon V \to \mathbb{R}$ such that

(a) I is a linear, i.e. $I(cf+g) = cI(f) + I(g)$ for all $f, g \in V$ and all $c \in \mathbb{R}$;

(b) I is nonnegative, i.e. if $f \in V$ and $f(\omega) \geq 0$ for all $\omega \in \Omega$, then $I(f) \geq 0$;

(c) $I(f_n) \downarrow 0$ whenever $f_n \in V$ and $f_n(\omega) \downarrow 0$ for all $\omega \in \Omega$ as $n \to \infty$.

REMARK 2.4.28 Condition (c) is clearly equivalent to each of the following conditions:

(c1) if $\{f_n\}_{n\geq 1} \subseteq V$ is an increasing sequence and there exists a function $f \in V$ such that

$$f \leq \lim_{n\to\infty} f_n,$$

then

$$I(f) \leq \lim_{n\to\infty} I(f_n),$$

and

(c2) if $\{f_n\}_{n\geq 1} \subseteq V$ are nonnegative functions and $g \in V$ such that

$$g \leq \sum_{n\geq 1} f_n,$$

then

$$I(g) \leq \sum I(f_n).$$

EXAMPLE 2.4.29 Let $V = C([0,1])$ and $I\colon V \to \mathbb{R}$ is the classical Riemann integral.
Clearly conditions (a) and (b) of Definition 2.4.27 are satisfied and condition (c) follows from Dini's theorem (see Theorem 1.6.28) and the properties of the Riemann integral.

Here is the main theorem to this alternative approach to integration. For a proof, we refer to Dudley (1989), p. 110 or Royden (1968), p. 297–299.

THEOREM 2.4.30 (Daniell-Stone Theorem)
If V is a vector lattice of $\mathbb{R}$-valued functions on a set Ω such that $f \wedge 1 \in V$ for all $f \in V$ and I is a Daniell integral on V, then there is a σ-field Σ of subsets of Ω and a measure μ on Σ such that

$$I(f) = \int_{\Omega} f \, d\mu,$$

for all $f \in V$.

PROPOSITION 2.4.31 If V is a vector lattice of $\mathbb{R}$-valued functions on a set Ω, $1 \in V$ and Σ is the smallest σ-field of subsets of Ω such that each $f \in V$ is σ-measurable, then for every Daniell functional I there is a unique measure μ on Σ such that

$$I(f) = \int_{\Omega} f \, d\mu,$$

for all $f \in V$.

REMARK 2.4.32 In general constant functions need not belong to the vector lattice V.
Any vector lattice V containing the constant functions is called a "Stone vector lattice".