# On exact solutions of a heat-wave type with logarithmic front for the porous medium equation

View the article online for updates and enhancements.

# IOP ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

# On exact solutions of a heat-wave type with logarithmic front for the porous medium equation

**A L Kazakov[1], A A Lempert[1], S S Orlov[2] and S S Orlov[1]**

[1]Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, Russia
[2]Irkutsk State University, Irkutsk, Russia

E-mail: `kazakov@icc.ru`, `lempert@icc.ru`

**Abstract.** The paper deals with a nonlinear second-order parabolic equation with partial derivatives, which is usually called "the porous medium equation". It describes the processes of heat and mass transfer as well as filtration of liquids and gases in porous media. In addition, it is used for mathematical modeling of growth and migration of population. Usually this equation is studied numerically like most other nonlinear equations of mathematical physics. So, the construction of exact solution in an explicit form is important to verify the numerical algorithms. The authors deal with a special solutions which are usually called "heat waves". A new class of heat-wave type solutions of one-dimensional (plane-symmetric) porous medium equation is proposed and analyzed. A logarithmic heat wave front is studied in details. Considered equation has a singularity at the heat wave front, because the factor of the highest (second) derivative vanishes. The construction of these exact solutions reduces to the integration of a nonlinear second-order ordinary differential equation (ODE). Moreover, the Cauchy conditions lead us to the fact that this equation has a singularity at the initial point. In other words, the ODE inherits the singularity of the original problem. The qualitative analysis of the solutions of the ODE is carried out. The obtained results are interpreted from the point of view of the corresponding heat waves' behavior. The most interesting is a damped solitary wave, the length of which is constant, and the amplitude decreases.

## 1. Introduction
We consider the nonlinear porous medium equation [1, 2] having the form

$$T_t = \text{div}(k\nabla T), \quad k = k(T)$$

in the case of $k = k_0 T^\sigma$, $\sigma \in \mathbb{R}^{>0}$

$$T_t = k_0 \text{div}(T^\sigma \nabla T). \tag{1}$$

Here $T = T(t, \mathbf{x})$ is an unknown function; $t$ is time, $\mathbf{x}$ is a vector of spatial variables. Operators div and $\nabla$ act on $\mathbf{x}$.

The existence of plane symmetry allows us to reduce (1) to one-dimensional equation

$$u_t = uu_{\rho\rho} + \frac{u_\rho^2}{\sigma}, \tag{2}$$

where $u = u(t, \rho)$ is a unknown function, depending on the time variable $t$ and space variable $\rho$.

$$u|_{\rho=\mathcal{F}(t)} = 0, \tag{3}$$

where $\rho - \mathcal{F}(t) = 0$ is a front of the heat wave, defined in the plane of the variables $(t, \rho)$.

One of the classes of solutions of the porous medium equation is the heat waves (wave filtering) propagating on cold (zero temperature) background with a finite velocity. Geometrically the heat-wave type solution consists of two surfaces (perturbed solution $u(t, x) \geq 0$ and zero temperature background $u \equiv 0$), which are continuously joined along some sufficiently smooth line $x = b(t)$, called the front. In the linear case such solutions are apparently, known, since the time of J. Fourier.

It was Ya.B. Zeldovich, who first obtained solutions of (1), in the form of a heat wave for nonlinear heat conductivity problem [3]. In [4] similar results were obtained for filtration problems by G.I. Barenblatt. Heat waves in the class of piecewise-analytic functions were first considered by A.F. Sidorov [5, 6]. Using the methodology [6] the existence and uniqueness theorems are proved for solutions of one-dimensional [7, 8] and two-dimensional [9] problems. The method of power series [10] for the porous medium equation with special boundary conditions is applied in [11].

To solve (2)-(3) we reduce it to the Cauchy problem for nonlinear ordinary differential equations of second order with a singularity at the highest derivative. Such solutions of nonlinear partial differential equations in the literature are called "the exact solutions" [12, 13].

## 2. Construction of Exact Solutions
We assume that the solution of (2), (3) has the following form

$$u(t, \rho) = \psi(t, \rho)w(\xi), \quad \xi = \xi(t, \rho), \tag{4}$$

where $w(\xi)$ is an one-variable function, $\psi(t, \rho)$ is an elementary function, $\xi_t \xi_\rho \neq 0$.

Substituting (4) to (2) we have

$$ww'' + \frac{1}{\sigma}(w')^2 + \left[ 2\left(\frac{1}{\sigma} + 1\right)\frac{\psi_\rho}{\psi\xi_\rho} + \frac{\xi_{\rho\rho}}{\xi_\rho^2} \right]ww' +$$
$$+ \left( \frac{1}{\sigma}\frac{\psi_\rho^2}{\psi^2\xi_\rho^2} + \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2} \right)w^2 - \frac{\xi_t}{\psi\xi_\rho^2}w' - \frac{\psi_t}{\psi^2\xi_\rho^2}w = 0. \tag{5}$$

In order that (5) becomes an ODE we should solve an overdetermined system of partial differential equations

$$\frac{\psi_\rho}{\psi\xi_\rho} = a_1, \ \frac{\xi_{\rho\rho}}{\xi_\rho^2} = a_2, \ \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2} = a_3, \ \frac{\xi_t}{\psi\xi_\rho^2} = a_4, \ \frac{\psi_t}{\psi^2\xi_\rho^2} = a_5, \tag{6}$$

where $a_l \in \mathbb{R}$, $l = \overline{1, 5}$.

**Lemma 1.** Let $a_1 = -2a_2$, $a_3 = 2a_2^2$, then

(i) system (6) is solvable;

(ii) Eq. (2) has an exact solution having form

$$u(t, \rho) = -\alpha^2 f'(t)v(\xi), \quad \xi = \frac{\rho}{\alpha} + f(t), \quad f(t) = \left[ \begin{array}{l} c_1 t + c_2, \\ \ln(c_1 t + c_2)^{-\omega}, \end{array} \right. \tag{7}$$

where $v(\xi) = -w(\xi)/a_4, a_4 \neq 0$;

(iii) $v(\xi)$ satisfies the following ODE

$$vv'' + \frac{(v')^2}{\sigma} + v' + kv = 0. \tag{8}$$

Here $k = 0$, if $f(t) = c_1 t + c_2$ or $k = 1/\omega$, if $f(t) = \ln(c_1 t + c_2)^{-\omega}$.

The validity of Lemma 1 follows from the results presented in [14].

In order the solution of Eq. (2) in the form (7) satisfies the boundary-value condition (3) we have to append condition $v|_{\xi=0} = 0$. Therefore $\mathcal{F}(t) = c_1 t + c_2$ or $\mathcal{F}(t) = \ln(c_1 t + c_2)^{\alpha\omega}$. In the first case we have a linear solution $u(t, \rho) = -\sigma c_1 (\rho - c_1 t - c_2)$.

Now we consider the second case in more detail. Eq. (8) becomes degenerate, because the highest derivative coefficient vanishes. To ensure it's solvability it is necessary to satisfy the condition $v'|_{\xi=0} = v_1 \in \{-\sigma, 0\}$.

So, Cauchy conditions for Eq. (8) are

$$v|_{\xi=0} = 0, \quad v'|_{\xi=0} = v_1 \in \{-\sigma, 0\}. \tag{9}$$

The solutions of (8)-(9) are concave functions (see Figure. 1). It is easy to see that the global maximum is $v = 0$ in the case $v_1 = 0$ (Figure 1b).
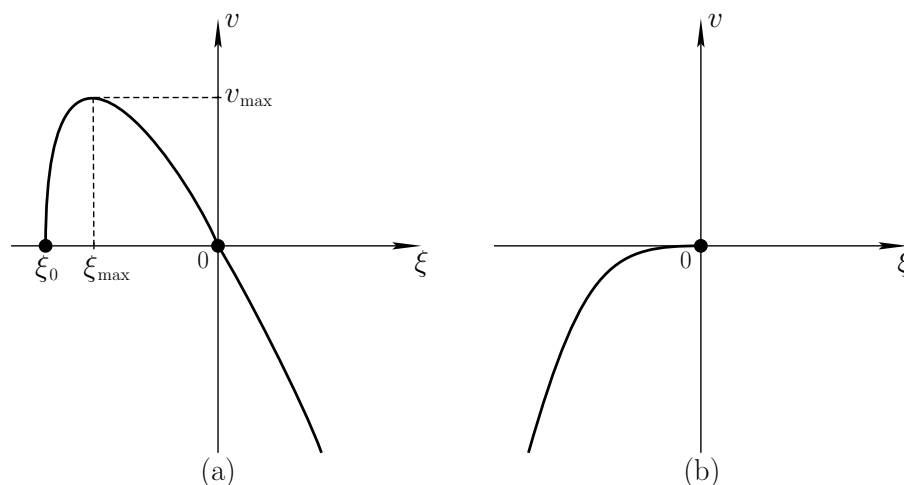


**Figure 1.** The behavior of the solution of (8)-(9): (a) for $v_1 = -\sigma$; (b) for $v_1 = 0$.

In the case $v_1 = -\sigma$ we have following Lemma.
**Lemma 2.** The following inequalities hold

$$-[\sqrt{\sigma(\sigma + 1)} + \sigma + 1]\omega \le \xi_0 \le -\sigma\omega,$$
$$-(\sigma + 1)\omega \le \xi_{\max} \le -\sigma\omega, \quad \frac{\sigma^2\omega}{2} \le v_{\max} \le \frac{\sigma(\sigma + 1)\omega}{2}.$$

## 3. On a heat wave with logarithmic front
Here we investigate the behavior of heat-wave type solutions with a logarithmic front. Let the heat wave moves from the origin to the right (to the first octant). This means that for function $\mathcal{F}(t) = \ln(c_1 t + c_2)^{\alpha\omega}$ (see (7)), which determines the law of the front movement following conditions are held

$$\mathcal{F}|_{t=0} = 0, \mathcal{F}'|_{t=0} \ge 0,$$

and $c_2 = 1$, $\alpha\omega c_1 > 0$. Then, in accordance with (7) we have

$$u(t,\rho) = \frac{\alpha^2\omega}{t + 1/c_1}\, v(\xi), \quad \xi = \frac{\rho}{\alpha} + \ln(c_1 t + 1)^{-\omega}. \tag{10}$$

Assuming (without loss of generality) that $\omega > 0$, we consider the two cases: $\omega > 0$, $\alpha > 0$, $c_1 > 0$ and $\omega > 0$, $\alpha < 0$, $c_1 < 0$.

**A.** Let $\omega > 0, \alpha > 0, c_1 > 0$. Obviously, in this case, if $v(\xi) \geq 0$, then $u(t,\rho) \geq 0$, therefore it is sufficient to consider the non-negative part of solution of (8) with respect to $v_1 = -\sigma$ (see Figure 1a). Because of $\xi_0 \leq \xi \leq 0$, then from (10) we obtain the field of heat wave propagation $\ln(c_1 t + 1)^{\alpha\omega} + \alpha\xi_0 \leq \rho \leq \ln(c_1 t + 1)^{\alpha\omega}$. Here we can see, that the heat wave has the back front $\rho = \ln(c_1 t + 1)^{\alpha\omega} + \alpha\xi_0$.
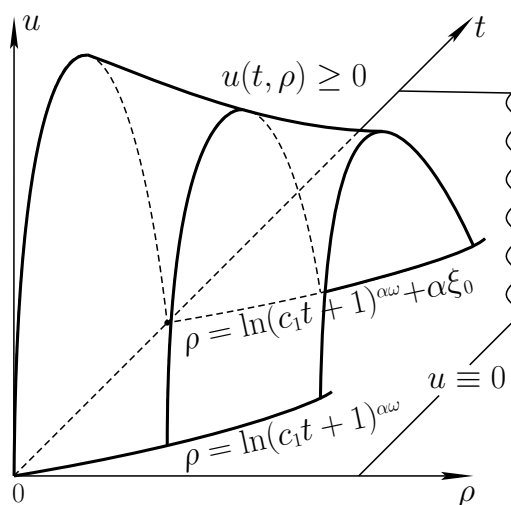


**Figure 2.** The heat wave with front $\rho = \ln(c_1 t + 1)^{\alpha\omega}$, $\alpha\omega > 0$, $c_1 > 0$.

Figure 2 shows schematically the behavior of a heat wave with a logarithmic front. The velocity of the main and back fronts decreases with time and tends to zero. From (10) and in view of the boundedness of $v(\xi)$ in $[\xi_0, 0]$ we obtain that $\lim_{t \to +\infty} u(t,\rho) = 0$ for each $\rho$ in the field of heat wave propagation. So, the complete cooling-down of the heated half-space occurs in an infinite time.

Using methods of power geometry [15, 16] we obtain the following relation

$$u(t,\rho) \underset{\xi \to 0}{\sim} -\frac{\alpha^2\sigma\omega}{t + 1/c_1}\left[\frac{\rho}{\alpha} + \ln(c_1 t + 1)^{-\omega}\right]. \tag{11}$$

It characterizes the behavior of the heat wave near the main front.

For any fixed $\rho$ the solution (10) has a maximum $u_{\max}$ at time $t_{\max}$. The inequalities below follow from (10) and Lemma 2.

$$0 \leq u_{\max} \leq \frac{\alpha^2\sigma(\sigma+1)\omega^2 c_1}{2}\, e^{-\frac{\rho}{\alpha\omega}}, \quad \frac{e^{\frac{\rho}{\alpha\omega}} - 1}{c_1} \leq t_{\max} \leq \frac{e^{\frac{\rho}{\alpha\omega}+\sigma+1} - 1}{c_1}.$$

**B.** Let $\omega > 0$, $\alpha < 0$, $c_1 < 0$. Then $0 \leq t < -1/c_1$ and $t = -1/c_1$ is the vertical asymptote of the front. According to (10) the inequality $u(t,\rho) \geq 0$ holds, if $v(\xi) \leq 0$. Consequently, we have the following subcases.

B1. We consider the nonpositive part of the solution $v(\xi)$ of Cauchy problem (8) for $v_1 = -\sigma$ (see Figure 1a). Here $\xi \geq 0$, so from (10) we obtain the domain $\rho \leq \ln(c_1 t + 1)^{\alpha\omega}$ where heat

wave exists. The heating of considered half-space to an arbitrary high temperature is carried out in finite time and there is no localization of heat. A similar situation corresponds to the HS blow-up regime [17], where $0 < -1/c_1 < +\infty$ is blow-up time (see Figure 3a). For any fixed $\rho$ the following relation holds

$$u(t,\rho) \underset{t \to -1/c_1 - 0}{\sim} -\frac{\alpha^2 \sigma \ln^2(c_1 t + 1)^\omega}{2(\sigma + 2)(t + 1/c_1)}.$$

The asymptotic formula (11) describes the heat wave behavior in the neighborhood of the front for $\xi \to 0$.

B2. Here we consider the nontrivial solution $v(\xi)$ of Cauchy problem (8) for $v_1 = 0$ (see Figure 1b). Here $\xi \le 0$, so from (10) we obtain the field of heat wave propagation $\rho \ge \ln(c_1 t + 1)^{\alpha\omega}$. Such heat wave can not be generated by any boundary regime if $\rho = 0$ and it is not a solution of the Sakharov problem (Figure 3b). However, it is easy to see that the constructed heat wave is a solution of some initial problem for Eq. (2). In this case, the solution tends to a cold background faster than any surface having form

$$u_n(t,\rho) = \frac{\alpha^2 \omega}{t + 1/c_1} \left[ \frac{\rho}{\alpha} + \ln(c_1 t + 1)^{-\omega} \right]^n, \ n \in \mathbb{N}.$$

For each fixed $t \in [0, -1/c_1)$, the following relation holds

$$u(t,\rho) \underset{\rho \to +\infty}{\sim} -\frac{\sigma \rho^2}{2(\sigma + 2)(t + 1/c_1)}.$$

A detailed proof is carried out by the methods of power geometry [15, 16] as well. It is quite cumbersome and will be the subject of a separate article.
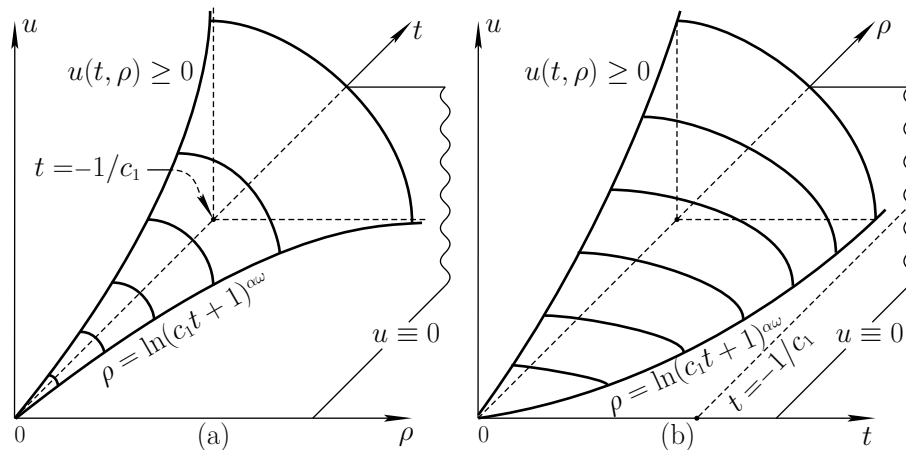


**Figure 3.** The heat wave with front $\rho = \ln(c_1 t + 1)^{\alpha\omega}$ for $\alpha\omega < 0$, $c_1 < 0$: (a) blow up regime; (b) without boundary regime.

## 4. Conclusion

Summarizing the research, it can be noted that the main purpose of the study is to construct and analyze special exact solutions of the one-dimensional porous medium equation. Constructing of exact solutions of nonlinear equations of mathematical physics is one of the most important problem in modern continuum mechanics. In particular, comparison of the results of calculations with exact solutions is one of the most effective ways of verifying numerical algorithms.

The proposed method for exact solutions constructing is based on the transformation to new dependent and independent variables. It usually aims to reduce the number of independent variables. Thus, in the case of two independent variables, the construction of exact solutions reduces to the integration of ordinary differential equations.

In this paper we address the case when the heat wave front moves in the plane of the variables $(\rho, t)$ according to the logarithmic law. The construction of the exact solution is reduced to the Cauchy problem for the second-order ordinary differential equation (ODE). This problem inherits a singularity of the original one, and therefore, it is solvable not for all values of the derivative at the initial point. The qualitative analysis of the solutions of the ODE is carried out. The obtained results are interpreted from the point of view the heat waves behavior. It is shown that the logarithmic law of the front moving corresponds to three possible configurations of the heat waves. The most interesting is a damped solitary wave, the length of which is constant, and the amplitude decreases.

Further research can be related both to the study of other laws of heat front movement (for example power or exponential), and to the consideration of cylindrical and spherical symmetries.

### References
[1] Vazquez J 2007 *The Porous Medium Equation: Mathematical Theory* (Oxford: Clarendon Press)
[2] Antontsev S and Shmarev S 2015 *Evolution PDEs with Nonstandard Growth Conditions* (Amsterdam: Atlantis Press)
[3] Zeldovich Y and Raizer Y 1966 *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena* vol 2 (New York: Academic Press)
[4] Barenblatt G, Entov V and Ryzhik V 1990 *Theory of Fluid Flow of Fluids Through Natural Rocks* (Dordrecht/Boston/London: Kluwer Academic Publ.)
[5] Vasin V and Sidorov A *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika.* **7** 13
[6] Sidorov A *Dokl. Akad. Nauk SSSR.* **280** 47
[7] Kazakov A and Lempert A *J. Appl. Mech. Tech. Phys.* **54** 97
[8] Kazakov A and Lempert A 2016 *J. Phys.: Conf. Series* **722** 012016
[9] Kazakov A and Kuznetsov P *J. Appl. Ind. Math.* **8** 227
[10] Teshukov V *Journal of Applied Mechanics and Technical Physics* **21** 261
[11] Furtak-Cole E, Telyakovskiy A and Cooper C 2013 *AGU Fall Meeting Abstracts*
[12] Galaktionov V and Svirshchevskii S 2006 *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics* (New-York: Chapman and Hall/CRC)
[13] Meleshko S 2005 *Methods for Constructing Exact Solutions of Partial Differential Equations* (New-York: Springer)
[14] Kazakov A, Lempert A, Orlov S and Orlov S *CEUR Workshop Proceedings* **1839** 344
[15] Bruno A 2000 *Power Geometry in Algebraic and Differential Equations* (Amsterdam: Elsevier Science)
[16] Bruno A *Russian Math. Surveys* **59** 429
[17] Samarskii A, Galaktionov V, Kurdyumov S and Mikhailov A 1995 *Blow-up in Quasilinear Parabolic Equations* (Berlin; New-York: de Gruyter)