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CONSTRUCTION AND STUDY OF EXACT SOLUTIONS TO A NONLINEAR HEAT EQUATION

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Abstract: We construct and study exact solutions to a nonlinear second order parabolic equation which is usually called the “nonlinear heat equation” or “nonlinear filtration equation” in the Russian literature and the “porous medium equation” in other countries. Under examination is the special class of solutions having the form of a heat wave that propagates through cold (zero) background with finite velocity. The equation degenerates on the boundary of a heat wave (called the heat front) and its order decreases. The construction of these solutions by passing to an overdetermined system and analyzing its solvability reduces to integration of nonlinear ordinary differential equations of the second order with an initial condition such that the equations are not solvable with respect to the higher derivative. Some admissible families of heat fronts and the corresponding exact solutions to the problems in question are obtained. A detailed study of the global properties of solutions is carried out by the methods of the qualitative theory of differential equations and power geometry which are adapted for degenerate equations. The results are interpreted from the point of view of the behavior and properties of heat waves with a logarithmic front.

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Introduction

A nonlinear heat equation is considered as one of the classical objects of the theory of partial differential equations and up to now remains a source of original ideas and new scientific results despite a long period of its study which is far from exhaustion. The interest in this equation is dictated not only by mathematics but also a wide range of its applications. It describes many processes that arise in heat and mass transfer, combustion and explosion, filtration, chemical kinetics, biology, etc.

In this article, we consider the equation

$$T_t = \operatorname{div}_{\mathbf{x}}(k(T)\operatorname{grad}_{\mathbf{x}} T), \quad k(T) = k_0 T^{\sigma}, \tag{PME}$$

where $T := T(t,\mathbf{x}) : \mathbb{R} \times \mathbb{R}^{\nu+1} \to \mathbb{R}$, $\nu \in \{0,1,2\}$, $\mathbb{R} \ni \sigma, k_0 > 0$, which is known in the literature as the nonlinear heat equation or the filtration equation [1] and also as the “porous medium equation” [2]. In dependence on the physical meaning, the quantity $T \geq 0$ stands for the temperature or the density of a medium and $k(T) \geq 0$ is either the heat conductivity coefficient or the filtration coefficient.

The authors consider the construction of the exact solutions to the equation (PME) having the form of a heat wave (of the heat wave type). By a heat wave we mean a configuration of the following two hypersurfaces: $T(t,\mathbf{x}) \geq 0$ (a perturbed solution) and $T(t,\mathbf{x}) \equiv 0$ (background trivial solution), continuously glued along a sufficiently smooth hypersurface $\Gamma(t,\mathbf{x}) = 0$ defining the heat wave front.

The description of the heat wave propagation in an absolutely cold medium (with zero background) with finite velocity and examples of solutions of the heat wave type for the first time were given in the article by Zeldovich and Kompaneets [3]. Close results for filtration problems in the study of different

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self-similar solutions were obtained shortly after by Barenblatt [4]. Later, the questions of solvability in the class of analytic functions of special boundary value problems for a heat equation having solutions of the heat wave type were considered in the articles by Sidorov and his followers. An original approach to the study of problems with a prescribed boundary regime (the Sakharov problem on initiation of a heat wave) based on the method of characteristic series was proposed and developed in [5]. By means of this methodology, a lot of theorems ensuring existence and uniqueness of locally analytic solutions to the above mentioned boundary value problems was proven in both the one-dimensional statement (see [6, 7]) (including the moving boundary case [8]) and in the multidimensional statement [9] as well.

We should note that despite the importance of the results of Sidorov's scientific school the question remains open of the behavior of solutions of the heat wave type beyond a small neighborhood, where the above-mentioned series diverge.

The exact solutions are more substantial in this case. First, their global behavior allows us to compose a general picture of behavior of the heat waves and second, using them we can illustrate different effects arising in the nonlinear heat conductivity (heat localization, etc.). In particular, the exact solutions (preferably traveling waves and self-similar solutions) serve as the basic model examples (see [10]) in the theory of blow-up regimes developed in the Samarskii scientific school.

In this article we obtain new classes of exact solutions to a nonlinear heat equation generating heat waves with power, exponential, and logarithmic fronts. Solutions are described with the help of functions being solutions to autonomous ordinary differential second order equations (generalized Liénard equations). The available analytic methods allows us to solve them in a finite form only in some particular cases. To study the behavior of solutions, we are going to involve dynamical systems for the purpose of the construction of phase portraits. Analysis of the phase plane is an accepted and most informative method of the study of exact solutions described by ordinary differential equations [11, Section 6.2; 12, Chapter 2]. Asymptotic expansions, based on the apparatus of the power geometry by Bruno [13, 14], essentially complement information on the properties of exact solutions. We study the properties of heat waves with a logarithmic law of the motion of the front in detail.

1. Statement of the Problem

The presence of space symmetries (plane, axial, spherical) allows us to reduce (PME) by means of nondegenerate transformations to the one-dimensional equation

$$u_t = uu_{\rho\rho} + \frac{u_\rho^2}{\sigma} + \frac{\nu u}{\rho}\, u_\rho, \tag{1.1}$$

where u is a new unknown function of time $t \geq 0$ and the nonnegative scalar variable $\rho := \|\mathbf{x}\|_\nu = \left(\sum_{k=1}^{\nu+1} x_k^2\right)^{1/2}$. The values of the parameter ν correspond to the heat propagation on the line ($\nu = 0$), the plane ($\nu = 1$), and in the space ($\nu = 2$) symmetrically with respect to the origin.

This article is devoted to the construction and qualitative study of the exact solutions to (1.1) of the heat wave type satisfying the boundary condition

$$u|_{\rho=\mathscr{F}(t)} = 0, \tag{1.2}$$

where $\rho = \mathscr{F}(t)$ is a given sufficiently smooth function on the plane of the variables (t, ρ) such that $\mathscr{F} \in C^k(\Omega)$, $\Omega \subset \mathbb{R}$, $1 \leq k \leq +\infty$. The curve defined by $\mathscr{F}$ is called the *front of a heat wave.* At its every point and at all other points with $u(t, \rho) = 0$, (1.1) degenerates with decrease of the order. The presence of this singularity relates (1.1) to nonclassical equations of mathematical physics that are topical now. Together with degenerate equations this class contains also equations not solvable with respect to the higher derivative which in dependence on the specifics are called in the literature the *Sobolev-type equations* [15, 16], or the *equations that are not of the Cauchy–Kowalevski-type* [17], and also *pseudoparabolic* [18].

2. Search of Exact Solutions

2.1. Construction of solutions. This section is devoted to finding the exact solutions to (1.1) which satisfy (1.2). We look for them in the form

$$u(t,\rho)=\psi(t,\rho)\theta(\xi),\quad \xi:=\xi(t,\rho),\quad \psi\xi_t\xi_\rho\not\equiv 0. \tag{2.1}$$

Representation (2.1) assumes the reduction to an ordinary differential equation with respect to $\theta(\xi)$. This method of construction of exact solutions is based on the Clarkson–Kruskal direct method [19]. To determine the functions ψ and ξ, we insert (2.1) into (1.1) and divide the relation obtained by the functional coefficient $\psi^2\xi_\rho^2\not\equiv 0$ at $\theta\theta''$ and $(\theta')^2$. Let us find sufficient conditions under which the equation

$$\theta\theta''+\frac{(\theta')^2}{\sigma}+\left\{\left[\frac{2(\sigma+1)}{\sigma}\frac{\psi_\rho}{\psi\xi_\rho}+\frac{\xi_{\rho\rho}}{\xi_\rho^2}+\frac{\nu}{\rho\xi_\rho}\right]\theta-\frac{\xi_t}{\psi\xi_\rho^2}\right\}\theta'$$
$$+\left(\frac{\psi_\rho^2}{\sigma\psi^2\xi_\rho^2}+\frac{\psi_{\rho\rho}}{\psi\xi_\rho^2}+\frac{\nu\psi_\rho}{\rho\psi\xi_\rho^2}\right)\theta^2-\frac{\psi_t}{\psi^2\xi_\rho^2}\,\theta=0$$

becomes an ordinary differential equation. Introduce the notation

$$\begin{gathered}\frac{\psi_\rho}{\psi\xi_\rho}=a_1,\quad \frac{\xi_{\rho\rho}}{\xi_\rho^2}=a_2,\quad \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2}=a_3,\quad \frac{\xi_t}{\psi\xi_\rho^2}=a_4,\\ \frac{\psi_t}{\psi^2\xi_\rho^2}=a_5,\quad \frac{\psi_\rho}{\rho\psi\xi_\rho^2}=a_6,\quad \frac{1}{\rho\xi_\rho}=a_7,\end{gathered} \tag{2.2}$$

where a_k are real constants, $a_4a_7\neq 0$, $k=\overline{1,7}$. Next, we need to solve the overdetermined system (2.2) of partial differential equations.

Lemma 1. *The system of differential equations* (2.2) *is solvable if and only if* $a_2=-a_1/2=-a_7$ *and* $a_2^2=a_3/2=a_6/2$.

Proof. The function $\xi(t,\rho)$ is explicitly defined from the second and seventh equations of system (2.2). We can make sure that these equations are consistent if and only if $a_2=-a_7\neq 0$. In this case the second, seventh, and fourth equations yield

$$\xi(t,\rho)=\log\left[\frac{\rho}{f(t)}\right]^{-\frac{1}{a_2}},\quad \psi(t,\rho)=\frac{\xi_t}{a_4\xi_\rho^2}=\frac{a_2[\log f(t)]'\rho^2}{a_4},$$

where $f(t)\not\equiv$ const in view of the condition $\xi_t\not\equiv 0$. Inserting $\xi(t,\rho)$ and $\psi(t,\rho)$ into the remaining equations of the system we obtain the relations $a_2=-a_1/2$, $a_2^2=a_3/2=a_6/2$ and the ordinary differential equation

$$ff''-\left(\frac{a_5}{a_2a_4}+1\right)(f')^2=0;$$

hence either $f(t)=c_2e^{c_1t}$ if $a_5=0$ or $f(t)=(c_1t+c_3)^{-a_2a_4/a_5}$ if $a_5\neq 0$, where $\mathbb{R}\ni c_1,c_2\neq 0$, $c_3\in\mathbb{R}$. □

The case of $\nu=0$ deserves a particular attention. As is easily seen, (2.2) for $\xi(t,\rho)$ and $\psi(t,\rho)$ takes the form

$$\frac{\psi_\rho}{\psi\xi_\rho}=a_1,\quad \frac{\xi_{\rho\rho}}{\xi_\rho^2}=a_2,\quad \frac{\psi_{\rho\rho}}{\psi\xi_\rho^2}=a_3,\quad \frac{\xi_t}{\psi\xi_\rho^2}=a_4,\quad \frac{\psi_t}{\psi^2\xi_\rho^2}=a_5. \tag{2.3}$$

Lemma 2. *The system of differential equations* (2.3) *is solvable if and only if* $a_2=-a_1/2$ *and* $a_2^2=a_3/2$.

Proof. We consider the two cases: $a_2\neq 0$ and $a_2=0$.

A. Let $a_2 \neq 0$. The second and fourth equations of (2.3) imply that

$$\xi(t,\rho) = \log\left[\frac{\rho}{f(t)} + g(t)\right]^{-\frac{1}{a_2}},$$

$$\psi(t,\rho) = \frac{\xi_t}{a_4\xi_\rho^2} = \frac{[\rho + g(t)f(t)]\{[\log f(t)]'\rho - g'(t)f(t)\}}{a_4},$$

where $f(t) \not\equiv 0$ and $[f'(t)]^2 + [g'(t)]^2 \not\equiv 0$ due to the condition $\xi_t\xi_\rho \not\equiv 0$.

If $g(t) \equiv 0$ then the proof is similar to that in Lemma 1.

Let $g(t) \not\equiv 0$. Inserting $\xi(t,\rho)$ and $\psi(t,\rho)$ into the first, third, and fifth equations of (2.3), and collecting the summands, we infer

$$\begin{gathered}(a_1 + 2a_2)f'\rho + a_2 gff' - (a_1 + a_2)g'f^2 = 0,\\ \big(2a_2^2 - a_3\big)f'\rho + 2a_2 gff' + a_3 g'f^2 = 0,\end{gathered} \tag{2.4}$$

$$\begin{aligned}&\left[ff'' - \left(\frac{a_5}{a_2a_4} + 1\right)(f')^2\right]\rho^2 + f^2\left(gf'' + \frac{2a_5}{a_2a_4}\,g'f' - g''f\right)\rho\\ &\qquad - f^3\left\{2gg'f' + \left[gg'' + \left(\frac{a_5}{a_2a_4} + 1\right)(g')^2\right]f\right\} = 0.\end{aligned} \tag{2.5}$$

To convert the expressions (2.4) and (2.5) into identities we equate the functional coefficients of the powers of ρ to zero. Equations (2.4) imply that

$$(a_1 + 2a_2)f' = \big(2a_2^2 - a_3\big)f' = 0,$$

and so $a_2 = -a_1/2$, $a_2^2 = a_3/2$, or $f(t) \equiv \text{const}$. The latter variant yields $g(t) \equiv \text{const}$ and so $[f'(t)]^2 + [g'(t)]^2 \equiv 0$; hence, we can exclude it from the consideration. In the former variant, we have $gf' + g'f = gf' + a_2 g'f = 0$. The case of $a_2 \neq 1$ is not of interest since it leads to $f(t)g(t) \equiv 0$ or $[f'(t)]^2 + [g'(t)]^2 \equiv 0$. For $a_2 = 1$, we obtain a system of four differential equations

$$gf' + g'f = 0, \quad ff'' - \left(\frac{a_5}{a_4} + 1\right)(f')^2 = 0,$$

$$gf'' + \frac{2a_5}{a_4}\,g'f' - g''f = 0, \quad 2gg'f' + \left[gg'' + \left(\frac{a_5}{a_4} + 1\right)(g')^2\right]f = 0,$$

which under the conditions $f(t)g(t) \not\equiv 0$ and $[f'(t)]^2 + [g'(t)]^2 \not\equiv 0$ possesses the properties

(1) if $a_5 = 0$ then $f(t) = c_2 e^{c_1 t}$ and $g(t) = c_3 e^{-c_1 t}$;

(2) if $a_5 \neq 0$ then $f(t) = (c_1 t + c_2)^{-a_4/a_5}$, $g(t) = (c_3/c_2)^{a_4/a_5}(c_1 t + c_2)^{a_4/a_5}$.

Here $\mathbb{R} \ni c_i \neq 0$, $i = 1, 2, 3$.

B. Let $a_2 = 0$. The second and fourth equations of (2.3) yield

$$\xi(t,\rho) = \frac{\rho}{g(t)} + f(t), \quad \psi(t,\rho) = \frac{\xi_t}{a_4\xi_\rho^2} = -\frac{g'\rho - g^2(t)f'(t)}{a_4},$$

where $g(t) \not\equiv 0$ and $[f'(t)]^2 + [g'(t)]^2 \not\equiv 0$, in view of the condition $\xi_t\xi_\rho \not\equiv 0$. Inserting $\xi(t,\rho)$ and $\psi(t,\rho)$ into the remaining equations, we infer $a_3 = 0$ and

$$a_1 g'\rho - a_1 g^2 f' - gg' = 0, \tag{2.6}$$

$$\frac{a_5}{a_4}\,(g')^2\rho^2 + g^2\left(g'' - \frac{2a_5}{a_4}\,g'f'\right)\rho - g^3\left\{g\left[f'' - \frac{a_5}{a_4}\,(f')^2\right] + 2g'f'\right\} = 0. \tag{2.7}$$

To convert (2.6) and (2.7) into identities we equate the functional coefficients of the powers of ρ to zero. Let $g(t) \equiv \text{const} \neq 0$ and let $a_1 \neq 0$ in (2.6). In this case $f(t) \equiv \text{const}$ and $[f'(t)]^2 + [g'(t)]^2 \equiv 0$, hence, this variant does not suits us. If we put $a_1 = 0$ in (2.6) then $g(t) \equiv \text{const} \neq 0$, and (2.7) turns into the differential equation

$$f'' - \frac{a_5}{a_4}(f')^2 = 0,$$

which implies that $f(t) = c_1 t + c_2$ whenever $a_5 = 0$ and $f(t) = \log(c_1 t + c_2)^{-a_4/a_5}$ whenever $a_5 \neq 0$, where $\mathbb{R} \ni c_1 \neq 0$ and $c_2 \in \mathbb{R}$. $\square$

Remark 1. Since $\psi(t,\rho) = \xi_t / \big(a_4 \xi_\rho^2\big)$, we assume for brevity that $\theta(\xi) = -a_4 v(\xi)$ in (2.1).

In the proof of Lemma 2 we have established that (1.1) admits the following classes of exact solutions:

A. For $\nu \in \{0, 1, 2\}$, we have

$$u(t,\rho) = \alpha[\log f(t)]'\rho^2\, v(\xi), \quad \xi = \log\left[\frac{\rho}{f(t)}\right]^{\frac{1}{\alpha}}. \tag{2.8}$$

Here either $f(t) = c_2 e^{c_1 t}$ or $f(t) = (c_1 t + c_3)^{\alpha\omega}$, where $\mathbb{R} \ni \alpha, \omega, c_1, c_2 \neq 0$, $c_3 \in \mathbb{R}$, and $v(\xi)$ satisfies the ordinary differential equation

$$vv'' + \frac{(v')^2}{\sigma} + \left\{\frac{\alpha[(\nu+3)\sigma + 4]}{\sigma}\, v + 1\right\} v' + \frac{2\alpha^2[(\nu+1)\sigma + 2]}{\sigma}\, v^2 + kv = 0. \tag{2.9}$$

Here $k = 0$ for $f(t) = c_2 e^{c_1 t}$; and $k = 1/\omega$ for $f(t) = (c_1 t + c_3)^{\alpha\omega}$.

B. For $\nu = 0$, we have the two additional classes of exact solutions. The former of them is of the form

$$u(t,\rho) = -[\rho + g(t)f(t)]\{[\log f(t)]'\rho - g'(t)f(t)\}v(\xi), \quad \xi = \log\left[\frac{\rho}{f(t)} + g(t)\right]^{-1}. \tag{2.10}$$

Here either $f(t) = c_2 e^{c_1 t}$ and $g(t) = c_3 e^{-c_1 t}$ or $f(t) = (c_1 t + c_2)^{-\omega}$ and $g(t) = (c_3 c_1 t / c_2 + c_3)^{\omega}$, where $\mathbb{R} \ni \omega, c_i \neq 0$, $i = 1, 2, 3$, and $v(\xi)$ satisfies the ordinary differential equation

$$vv'' + \frac{(v')^2}{\sigma} - \left(\frac{3\sigma + 4}{\sigma}\, v - 1\right) v' + \frac{2(\sigma + 2)}{\sigma}\, v^2 + kv = 0. \tag{2.11}$$

Here $k = 0$ for $f(t) = c_2 e^{c_1 t}$, $g(t) = c_3 e^{-c_1 t}$ and $k = 1/\omega$ for $f(t) = (c_1 t + c_2)^{-\omega}$, $g(t) = (c_3 c_1 t / c_2 + c_3)^{\omega}$.

Remark 2. Equation (2.9) for $\nu = 0$ and $\alpha = -1$ agrees with (2.11).

The other class of exact solutions is defined as follows:

$$u(t,\rho) = -\alpha^2 f'(t) v(\xi), \quad \xi = \frac{\rho}{\alpha} + f(t). \tag{2.12}$$

Here either $f(t) = c_1 t + c_2$ or $f(t) = \log(c_1 t + c_2)^{-\omega}$, where $\mathbb{R} \ni \alpha, \omega, c_1 \neq 0$, $c_2 \in \mathbb{R}$, and $v(\xi)$ satisfies the ordinary differential equation

$$vv'' + \frac{(v')^2}{\sigma} + v' + kv = 0. \tag{2.13}$$

In this case $k = 0$ if $f(t) = c_1 t + c_2$ or $k = 1/\omega$ if $f(t) = \log(c_1 t + c_2)^{-\omega}$.

2.2. Agreement conditions of a solution and the boundary condition. To find solutions to (1.1) of the heat wave type, we need to choose among the above solutions the exact solutions satisfying (1.2).

To agree exact solutions (2.8), (2.10), and (2.12) with (1.2), it suffices to assume that $v|_{\xi=\xi_0} = 0$, where $\xi_0 \in \mathbb{R}$. In this case, since the differential equations for $v(\xi)$ are autonomous, without loss of generality we can put $\xi_0 = 0$. Indeed, the condition $v|_{\xi=0} = 0$ leads to the situation that the integral surfaces defined by the solutions (2.8), (2.10), and (2.12) to (1.1) intersect the plane $u \equiv 0$ along some curves $\rho = \mathscr{F}(t)$. They are

(1) $\rho = c_2 e^{c_1 t}$ or $\rho = (c_1 t + c_3)^{\alpha\omega}$ for (2.8);

(2) $\rho = c_2(e^{c_1 t} - c_3)$ or $\rho = (c_1 t + c_2)^{-\omega} - (c_2/c_3)^{-\omega}$ for (2.10);

(3) $\rho = c_1 t + c_2$ or $\rho = \log(c_1 t + c_2)^{\alpha\omega}$ for (2.12).

The curves defined by the above functions are the fronts of the corresponding heat waves.

Thus, solutions to the ordinary differential equations (2.9), (2.11), and (2.13), satisfying the initial condition

$$v|_{\xi=0} = 0, \quad v'|_{\xi=0} = v_1, \tag{2.14}$$

give rise solutions of the heat wave type to the initial equation (1.1) with partial derivatives. Moreover, the cases $v_1 = -\sigma$ or $v_1 = 0$ conform with equations (2.9), (2.11), and (2.13).

We should note that it is probably impossible to find general integrals of each of the equations (2.9), (2.11), and (2.13) for all values of parameters. Hence, in order to establish behavior and properties of solutions which are of interest, we need additional studies. Before the proper analysis, we describe several cases in which we can find an explicit form of the corresponding exact solutions.

Remark 3. The trivial solution $v(\xi) \equiv 0$, which obviously is a solution to (2.9), (2.11), and (2.13), is excluded from considerations.

2.3. The cases solvable in finite form. Equations (2.9), (2.11), and (2.13) belong to the class of generalized Liénard equations. Finding the finite solvability conditions for these equations are conducted by Kudryashov (see, for instance, [20]). We point out the cases that we can write out the explicit form of the exact solutions to (1.1), (1.2).

A. Let $\nu = 0$ and $\mathscr{F}(t) = c_1 t + c_2$. The boundary condition (1.2) is satisfied for solutions (2.8), (2.10), and (2.12), in which $v(\xi) = -\sigma\omega(e^{\xi/\omega} - 1)/e^{2\xi/\omega}$, $v(\xi) = -\sigma e^{\xi}(e^{\xi} - 1)$, and $v(\xi) = -\sigma\xi$, respectively, i.e., the Cauchy problem for (2.9), (2.11), and (2.13) with the initial condition (2.14) for $v_1 = -\sigma$ can be integrated in the explicit form.

It is easy to make sure that we can obtain a linear solution to the problem (1.1), (1.2), of the form $u(t,\rho) = -\sigma c_1(\rho - c_1 t - c_2)$, which for $c_1 > 0$ is a heat wave on the set $t \geq 0$, $0 \leq \rho \leq c_1 t + c_2$.

B. Let $\nu = 0$ and $\mathscr{F}(t) = (c_1 t + c_2)^{1/(\sigma+2)} - (c_2/c_3)^{1/(\sigma+2)}$. A solution (2.10) satisfies the boundary condition (1.2). Solving the Cauchy problem (2.11), (2.14) for $v_1 = -\sigma$, we find that $v(\xi) = -\sigma(e^{2\xi} - 1)/2$ and so (2.10) takes the form

$$u(t,\rho) = -\frac{\sigma c_1}{2(\sigma+2)(c_1 t + c_2)}\left\{\left[\rho + \left(\frac{c_2}{c_3}\right)^{\frac{1}{\sigma+2}}\right]^2 - (c_1 t + c_2)^{\frac{2}{\sigma+2}}\right\}.$$

Under the conditions $c_1 > 0$, $c_2 > 0$, and $c_3 \geq 1$, a solution **B** is a heat wave on the set $t \geq 0$, $0 \leq \rho \leq (c_1 t + c_2)^{1/(\sigma+2)} - (c_2/c_3)^{1/(\sigma+2)}$.

C. Let $\nu = 0$ and $\mathscr{F}(t) = (c_1 t + c_2)^{1/[2(\sigma+1)]} - (c_2/c_3)^{1/[2(\sigma+1)]}$. The solution (2.10) satisfies the boundary condition (1.2). Solving the Cauchy problem (2.11), (2.14) for $v_1 = -\sigma$, we infer

$$v(\xi) = -\sigma(\sigma+1)[e^{(\sigma+2)\xi/(\sigma+1)} - 1]/(\sigma+2),$$

and, hence, (2.10) takes the form

$$u(t,\rho) = -\frac{\sigma c_1}{2(\sigma+2)(c_1 t + c_2)}\left[\rho + \left(\frac{c_2}{c_3}\right)^{\frac{1}{2(\sigma+1)}}\right]^{\frac{\sigma}{\sigma+1}}$$
$$\times\left\{\left[\rho + \left(\frac{c_2}{c_3}\right)^{\frac{1}{2(\sigma+1)}}\right]^{\frac{\sigma+2}{\sigma+1}} - (c_1 t + c_2)^{\frac{\sigma+2}{2(\sigma+1)^2}}\right\}.$$

Under the conditions $c_1 > 0$, $c_2 > 0$, and $c_3 \geq 1$, the solution **C** is a heat wave on the set $t \geq 0$, $0 \leq \rho \leq (c_1 t + c_2)^{1/(2\sigma+2)} - (c_2/c_3)^{1/(2\sigma+2)}$.

D. Let $\mathscr{F}(t) = (c_1 t + c_2)^{1/[(\nu+1)\sigma+2]}$. The solution (2.8) satisfies the boundary condition (1.2). Solving the Cauchy problem (2.9), (2.14) for $v_1 = -\sigma$, we infer that

$$v(\xi) = \frac{[(\nu+1)\sigma+2]\sigma\omega}{2}(e^{\frac{-2\xi}{[(\nu+1)\sigma+2]\omega}} - 1),$$

and, hence, (2.8) takes the form

$$u(t,\rho) = -\frac{\sigma c_1}{2[(\nu+1)\sigma+2](c_1 t + c_2)}[\rho^2 - (c_1 t + c_2)^{\frac{2}{(\nu+1)\sigma+2}}].$$

Under the conditions $c_1 > 0, c_2 > 0$, the solution **D** is a heat wave on the set $t \geq 0$, $0 \leq \rho \leq (c_1 t + c_2)^{1/[(\nu+1)\sigma+2)]}$.

E. Let $\mathscr{F}(t) = (c_1 t + c_2)^{1/[2(\sigma+1)]}$. A solution (2.8) satisfies the boundary condition (1.2). Solving the Cauchy problem (2.9), (2.14) for $v_1 = -\sigma$, we infer

$$v(\xi) = \frac{2(\sigma+1)^2\sigma\omega}{(\nu+1)\sigma+2}(e^{\frac{-[(\nu+1)\sigma+2]\xi}{2(\sigma+1)^2\omega}} - 1),$$

and, hence, (2.8) takes the form

$$u(t,\rho) = -\frac{\sigma c_1 \rho^{-\frac{(\nu-1)\sigma}{2(\sigma+1)}}}{2[(\nu+1)\sigma+2](c_1 t + c_2)}[\rho^{\frac{(\nu+1)\sigma+2}{\sigma+1}} - (c_1 t + c_2)^{\frac{(\nu+1)\sigma+2}{2(\sigma+1)^2}}].$$

Under the conditions $c_1 > 0$ and $c_2 > 0$ the solution **E** is a heat wave on the set $t \geq 0$, $0 < \rho \leq (c_1 t + c_2)^{1/(2\sigma+2)}$.

We can see that, for $\nu = 1$, the solutions **D** and **E** coincide. Note that the above exact solutions to (1.1) for $\nu = 0$ agrees with the available formulas in [21, p. 216].

3. Study of Ordinary Differential Equations

The exact solutions to (1.1) of the heat wave type (i.e. satisfying (1.2)), constructed in Section 2, contain solutions to different Cauchy problems to nonlinear ordinary second order differential equations as factors and they are not integrable in quadratures except for some particular cases. In this section we make some qualitative analysis of these equations.

Using a unified approach, we can rewrite (2.9), (2.11), and (2.13) as

$$vv'' + \frac{(v')^2}{\sigma} + (K_1 v + 1)\, v' + K_2 v^2 + K_3 v = 0. \tag{3.1}$$

Here $K_i \in \mathbb{R}$, $i = 1, 2, 3$, $K_2 \geq 0$, and, moreover, $K_1 K_2 \neq 0$ (see (2.9), (2.11)) or $K_1^2 + K_2^2 = 0$ (see (2.13)).

Since (3.1) is autonomous, it is convenient to consider the phase plane (v, v').

3.1. Passage to a dynamical system. To (3.1), there corresponds the dynamical system

$$\frac{dv}{d\xi} = w, \quad \frac{dw}{d\xi} = -\frac{w^2/\sigma + (K_1 v + 1)\, w + K_2 v^2 + K_3 v}{v}. \tag{3.2}$$

Note that the right-hand side of the second equation in (3.2) is not defined at the points $(0, w)$ which are of interest from the standpoint of our aims (see (2.14)). Represent (3.2) as

$$\frac{dv}{d\xi} = \frac{P(v,w)}{R(v,w)}, \quad \frac{dw}{d\xi} = \frac{Q(v,w)}{R(v,w)},$$

where $P(v,w) = vw$, $Q(w,v) = -w^2/\sigma - (K_1 v + 1)w - K_2 v^2 - K_3 v$, $R(v,w) = v$, and change its parametrization putting $d\xi = R(w,v)\, d\zeta$. As a result, we obtain

$$\frac{dv}{d\zeta} = P(v,w), \quad \frac{dw}{d\zeta} = Q(v,w), \tag{3.3}$$

thereby redefining (3.2) at $(0, w)$ by continuity. Obviously, in any part of the domain $G \subset \mathbb{R}^2$ in which $v \neq 0$, the trajectories of (3.2) and (3.3) coincide as point sets; however, their parameters are different. In this case, the directions ζ and ξ agree if $v > 0$ and they are opposite if $v < 0$. It is naturally to distinguish the points at which $v = 0$ and to assume that they do not belong to trajectories of (3.3) [22, Chapter 1, Section 8].

Thus, we have passed from (3.1) to the dynamical system (3.3) whose phase portrait will allow us to describe the behavior of solutions to (3.1).

3.2. Character of critical points. We begin with analysis of critical points. Find and study equilibria (critical points) of (3.3). The condition $P(v,w) = Q(v,w) = 0$ gives

(1) the three equilibria $(0, -\sigma)$, $(0, 0)$, and $(-K_3/K_2, 0)$ whenever $K_1 K_2 K_3 \neq 0$;

(2) the two equilibria $(0, -\sigma)$ and $(0, 0)$ whenever $K_1 = K_2 = 0$, $K_3 \neq 0$ or $K_1 K_2 \neq 0$, $K_3 = 0$;

(3) the equilibrium $(0, -\sigma)$ whenever all $K_i = 0$.

Next, we use the following notations:

$$M(v,w) := \left\| \begin{matrix} P_v & P_w \\ Q_v & Q_w \end{matrix} \right\|, \quad \Delta(v,w) := \det M, \quad \delta(v,w) := P_v + Q_w.$$

In our case we have

$$\Delta(v,w) = -\frac{2w^2}{\sigma} - w + 2K_2 v^2 + K_3 v, \quad \delta(v,w) = \frac{(\sigma - 2)w}{\sigma} - K_1 v - 1.$$

Study the character of every critical point separately.

A. Consider the point $(0, -\sigma)$. We have $\Delta|_{v=0,\, w=-\sigma} = -\sigma \neq 0$; hence, it is a simple equilibrium. The roots of the characteristic equation

$$\det(M - \lambda E)|_{v=0, w=-\sigma} = (\lambda + \sigma)(\lambda - 1) = 0$$

are the numbers $\lambda_1 = -\sigma$ and $\lambda_2 = 1$. Since in this case $\Delta < 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\lambda_1 \lambda_2 < 0$, the topological type of the point $(0, -\sigma)$ is a saddle.

B. Consider the point $(0, 0)$ for which $\Delta|_{v=w=0} = 0$; hence, the equilibrium is complex. To study it, we use the technique described in [22, Chapter 4]. Here $\delta|_{v=w=0} = -1 \neq 0$ and the equation resulting from (3.3) (we omit some transformations) is representable as

$$P^*(v,w)\, dw - [kw + Q^*(v,w)]\, dv = 0,$$

where $P^*(v,w) = vw$, $Q^*(v,w) = -w^2/\sigma - K_1 vw - K_2 v^2 - K_3 v$, $k = -1$.

Consider the cases $K_3 \neq 0$ and $K_3 = 0$ separately.

B1. Let $K_3 \neq 0$. Represent a solution to the equation $kw + Q^*(v,w) = 0$ in the form of a series in the powers of v. Inserting it in $P^*(v,w)$, we obtain

$$w = \varphi(v) = -K_3 v + \cdots, \quad \psi(v) = P^*|_{w=\varphi(v)} = -K_3 v^2 + \cdots.$$

Since the least power of v in the expansion $\psi(v)$ is equal to 2 (even), the equilibrium $(0,0)$ is a saddle-node with one nodal sector and two saddle sectors. We should note that since $k < 0$, the nodal sector is stable. Moreover, if $K_3 < 0$ then the trajectories of the nodal sector tend to $(0,0)$ on the left of the axis Ow as $\zeta \to -\infty$ and if $K_3 > 0$ then they converge on the right of the axis Ow as $\zeta \to +\infty$.

B2. Let $K_3 = 0$. Then the expansions hold:

$$w = \varphi(v) = lv^2 + \cdots, \quad \psi(v) = P^*|_{w=\varphi(v)} = lv^3 + \cdots,$$

where $l = (\sigma - 1)K_1^2/8 - K_2/2$. The least power of v in the expansion of $\psi(v)$ is equal to 3 (odd); hence, for $l < 0$ the equilibrium $(0,0)$ is a complex node (stable due to the inequality $k < 0$) and for $l > 0$ it is a complex saddle.

C. Consider the point $(-K_3/K_2, 0)$. In this case $\Delta|_{v=-K_3/K_2, w=0} = K_3^2/K_2 \neq 0$; hence, it is a simple equilibrium. Represent the characteristic equation

$$\det(M - \lambda E)|_{v=-K_3/K_2, w=0} = K_2\lambda^2 - (K_1K_3 - K_2)\lambda + K_3^2 = 0$$

in the general form $\lambda^2 - 2p\lambda + q = 0$, where $p = (K_1K_3 - K_2)/(2K_2)$, and $q = K_3^2/K_2$. Its roots are $\lambda_{1,2} = p \pm \sqrt{p^2 - q}$. In dependence of the values of p and q, the equilibrium $(-K_3/K_2, 0)$ can be of the following qualitative character: a focus (stable for $-\sqrt{q} < p < 0$ and unstable for $0 < p < \sqrt{q}$); a node (stable for $p < -\sqrt{q}$ and unstable for $p > \sqrt{q}$). Since $\lambda_1\lambda_2 = q > 0$, this point is not a saddle.

The cases $p = 0$ and $p = \pm\sqrt{q}$ (for a linearized system the center and a degenerate node, respectively) in view of structural instability of topological types of these points require an additional study which is beyond the framework of this article.

4. Study of Heat Waves with a Logarithmic Front

To demonstrate effectiveness of the above approach, using the results obtained we study solutions of the form (2.12) in detail; they define heat waves with a logarithmic law of the motion of a heat front (with a logarithmic front).

4.1. Existence and uniqueness theorems. Consider solutions of the heat wave type with a logarithmic front $\rho = \log(c_1 t + c_2)^{\alpha\omega}$ obtained form (2.12). Recall that the function $v(\xi)$ is defined in this case as a solution to the Cauchy problem

$$\begin{gathered} vv'' + \frac{(v')^2}{\sigma} + v' + \frac{v}{\omega} = 0, \\ v|_{\xi=0} = 0, \quad v'|_{\xi=0} = v_1 \in \{-\sigma, 0\}. \end{gathered} \tag{4.1}$$

Without loss of generality, we can assume that $\omega > 0$, since solutions to (4.1) possess the symmetry $v(\xi, -\omega) = -v(-\xi, \omega)$, i.e., the change of the sign of ω is equivalent to the rotation of the graphics of solutions by 180° about the origin O.

As it is often the case in the study of complicated nonlinear equations and systems, the authors could not find a finite form of solutions (different from trivial) to (4.1) with the use of known analytic methods. Nevertheless, we can describe the behavior of these solutions. To this end, we employ the results of Section 3. By analogy to the above arguments, we pass to the dynamical system

$$\frac{dv}{d\xi} = w, \quad \frac{dw}{d\xi} = -\frac{w^2/\sigma + w + v/\omega}{v}. \tag{4.2}$$

Redefining (4.2) with the use of the change $d\xi = v\,d\zeta$ by continuity at the points $(0,\, w)$, we arrive at the system

$$\frac{dv}{d\zeta} = vw, \quad \frac{dw}{d\zeta} = -\frac{w^2}{\sigma} - w - \frac{v}{\omega}, \tag{4.3}$$

for which $(0, -\sigma)$ is a saddle and $(0, 0)$ is a saddle-node (see Section 3.2, B1). Both points possess vertical semiseparatrices lying on the axis Ow. Also there are three nonvertical semiseparatrices s_1, s_2, and s_3 (Fig. 1(a)); moreover, s_1 and s_2 tend to $(0, -\sigma)$ as $\zeta \to -0$ with the slope $1/[(\sigma + 2)\omega]$, and s_3 tends to $(0, 0)$ as $\zeta \to +0$ with the slope $-1/\omega$.

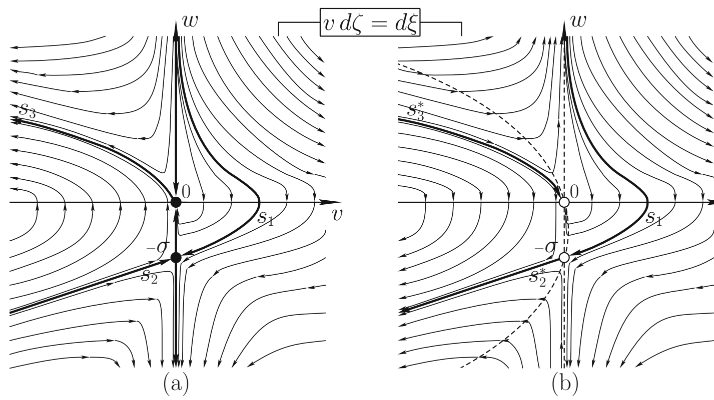

Fig. 1. The phase portrait: (a) systems (4.3); (b) systems (4.2)

The inverse passage to system (4.2) (with the initial parametrization) excludes all points of the form $(0, w)$ and changes the directions of trajectories lying in the left half-plane ($v < 0$) on opposite. In particular, s_2 transforms into the trajectory s_2^*, which tends to $(0, -\sigma)$ as $\xi \to +0$ with slope $1/[(\sigma+2)\omega]$ and s_3 transforms into the trajectory s_3^* tending to $(0, 0)$ as $\xi \to -0$ with the slope $-1/\omega$ (Fig. 1(b)).

Remark 4. The trajectories $s_1 \cup s_2^*$ and s_3^* are the only trajectories on the phase portrait of (4.2) tending to the points $(0, -\sigma)$ and $(0, 0)$; hence, they correspond to solutions to (4.1).

Thus, the form of the phase trajectories $s_1 \cup s_2^*$ and s_3^* allows us to describe the behavior of solutions to the Cauchy problem (4.1) as follows: $s_1 \cup s_2^*$ corresponds to a solution for $v_1 = -\sigma$ (Fig. 2(a)) and s_3^* corresponds to a nontrivial solution for $v_1 = 0$ (Fig. 2(b)).

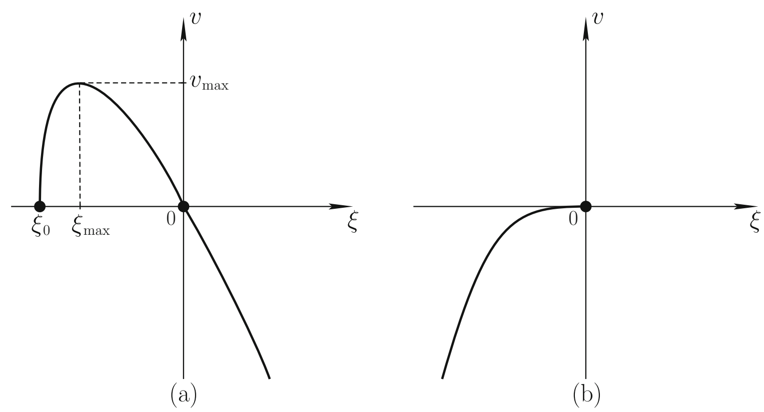

Fig. 2. The graphic of a solution to (4.1): (a) for $v_1 = -\sigma$; (b) for $v_1 = 0$

REMARK 5. A solution $v(\xi)$ to (4.1) for $v_1 = -\sigma$ cannot be extended on the left of the point $\xi_0 < 0$; moreover, $v|_{\xi=\xi_0} = 0$ and $\lim_{\xi\to\xi_0+0} v'(\xi) = +\infty$.

REMARK 6. A nontrivial solution $v(\xi)$ to (4.1) for $v_1 = 0$ cannot be extended on the right of the point 0; moreover, $v|_{\xi=0} = 0$ and $\lim_{\xi\to-0} v'(\xi) = 0$.

The arguments of this section validate the following theorem.

Theorem. *The Cauchy problem* (4.1) *in the case of*
(a) $v_1 = -\sigma$ *has a unique solution* $v(\xi) \in C_{[\xi_0,+\infty)} \cap C^2_{(\xi_0,+\infty)}$;
(b) $v_1 = 0$ *has a unique nontrivial solution* $v(\xi) \in C_{(-\infty,0]} \cap C^2_{(-\infty,0)}$.

REMARK 7. It is shown in [23] that a solution to (4.1) for $v_1 = -\sigma$ is analytic in some neighborhood about $\xi = 0$.

4.2. Interval estimates of solutions. We now prove some quantitative estimates of solutions to problem (4.1).

The location of the phase trajectories in question relative to the line defined by the equation $w^2/\sigma + w + v/\omega = 0$, and the coordinate axis Ow (they are displayed in Fig. 1(b) (dash-line)) is such that the second equation of (4.2) implies the inequality $dw/d\xi < 0$ and, therefore, $d^2v/d\xi^2 < 0$. Thus, solutions to (4.1) are concave (upper convex) functions. In particular, the global maximum of a solution to (4.1) for $v_1 = 0$ is equal to $v = 0$ and is achieved at $\xi = 0$.

Given the characteristic points ξ_0, $\xi_{\max}$ and the value $v_{\max}$ of a solution to (4.1) with $v_1 = -\sigma$, the following interval estimates are valid.

Lemma 3. *The inequalities hold:*

$$\begin{gathered} -[\sqrt{\sigma(\sigma+1)} + \sigma + 1]\omega \le \xi_0 \le -\sigma\omega, \\ -(\sigma+1)\omega \le \xi_{\max} \le -\sigma\omega, \quad \frac{\sigma^2\omega}{2} \le v_{\max} \le \frac{\sigma(\sigma+1)\omega}{2}. \end{gathered} \tag{4.4}$$

PROOF. Make the affine transformation

$$l : (\xi, v) \to ((\sigma+1)\omega\xi,\ \sigma(\sigma+1)\omega v).$$

As a result, the Cauchy problem (4.1) takes the form

$$\begin{gathered} vv'' + \frac{(v')^2}{\sigma} + \frac{v'}{\sigma} + \frac{(\sigma+1)v}{\sigma} = 0, \\ v|_{\xi=0} = 0, \quad v'|_{\xi=0} = -1, \end{gathered} \tag{4.5}$$

and the inequalities in question follows from the analysis of (4.5).

For $\xi \in [\xi_{\max}, 0]$, the function v decreases and $-1 \le v' \le 0$. Taking this estimate into account, we obtain from (4.5) that $v'' \ge -(\sigma+1)/\sigma$; moreover, the equality $v''|_{\xi=\xi_{\max}} = -(\sigma+1)/\sigma$ is achieved at $\xi = \xi_{\max}$. Subsequently integrating the estimate for v'' over the segment $[\xi, 0]$, we infer

$$v' \le -\frac{(\sigma+1)\xi}{\sigma} - 1, \quad v \ge -\frac{(\sigma+1)\xi^2}{2\sigma} - \xi, \quad \xi \in [\xi_{\max}, 0].$$

The first inequality for $\xi = \xi_{\max}$ validates the estimate $\xi_{\max} \le -\sigma/(\sigma+1)$ from above. The right-hand side of the second equality achieves its maximum at $\xi = -\sigma/(\sigma+1) \in [\xi_{\max}, 0)$ and it satisfies the inequality $v|_{\xi=-\sigma/(\sigma+1)} \ge \sigma/[2(\sigma+1)]$. Since $v|_{\xi=-\sigma/(\sigma+1)} \le v_{\max}$, we finally have the lower estimate $v_{\max} \ge \sigma/[2(\sigma+1)]$.

Studying equation (4.5), we can show that $v'' \le -1$ for $\xi \in (\xi_0, 0]$. Next, subsequently integrating over $[\xi, 0]$, we derive that

$$v' \ge -\xi - 1, \quad v \le -\frac{\xi^2}{2} - \xi, \quad \xi \in [\xi_{\max}, 0].$$

These inequalities justify the respective estimates $\xi_{\max} \geq -1$ and $v_{\max} \leq 1/2$.

For $\xi \in [\xi_0, \xi_{\max}]$, the function v increases, i.e., $v' \geq 0$. In view of this inequality, (4.5) implies that $v'' \leq -(\sigma+1)/\sigma$. Its subsequent twofold integration over $[\xi, \xi_{\max}]$ ensures that

$$v_{\max} - v \geq \frac{(\sigma+1)(\xi_{\max}-\xi)^2}{2\sigma}, \quad \xi \in [\xi_0, \xi_{\max}].$$

Since $\xi_0 < \xi_{\max}$ and $v_{\max} \leq 1/2$, for $\xi = \xi_0$, the last inequality takes the form $0 < \xi_{\max} - \xi_0 \leq \sqrt{\sigma/(\sigma+1)}$; thereby, $\xi_0 \to \xi_{\max}$ as $\sigma \to 0$ and $-\sqrt{\sigma/(\sigma+1)} - 1 \leq \xi_0 \leq -\sigma/(\sigma+1)$.

Applying the transformation l^{-1} to ξ_0, $\xi_{\max}$, and $v_{\max}$, we arrive at (4.4). $\square$

4.3. Asymptotic behavior of solutions. To find asymptotics and expansions of solutions to the Cauchy problem (4.1), we use the methods of power geometry [13]. In what follows, we omit some part of the most cumbersome calculations for defining critical numbers and the supports of the expansions of solutions of shortened equations (they are similar to those in [14]).

Introduce the notation

$$f(\xi, v) := vv'' + \frac{(v')^2}{\sigma} + v' + \frac{v}{\omega} = 0. \tag{4.6}$$

The Newton polygon $\Gamma(f)$ and the normal cones $\mathbf{U}_j^{(i)}$ of the generalized faces $\Gamma_j^{(i)}$, $i = 0, 1$, $j = 1, 2, 3$, of equation (4.6) are exhibited in Figs. 3(a) and 3(b), respectively.

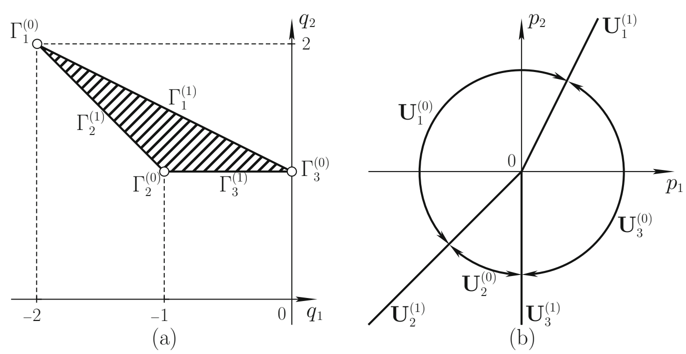

Fig. 3. The Newton polygon (a) and the normal cones (b) of (4.6)

A. To the vertex $\Gamma_1^{(0)}$, there correspond the shortened equation

$$\hat{f}_1^{(0)}(\xi, v) := vv'' + \frac{(v')^2}{\sigma} = 0 \tag{4.7}$$

and the normal cone $\mathbf{U}_1^{(0)} = \{(p_1 < 0,\ p_2/p_1 < 1) \cup (p_1 > 0,\ p_2/p_1 > 2)\}$. Inserting $v(\xi) = c_r \xi^r$ into (4.7), where $\mathbb{R} \ni c_r \neq 0$, we find that $r_1 = 0$ and $r_2 = \sigma/(\sigma+1)$. Thus, (4.7) has two power solutions $v_1(\xi) = c_0$ and $v_2(\xi) = c_{\frac{\sigma}{\sigma+1}} \xi^{\frac{\sigma}{\sigma+1}}$. Since only one vector $-(1, r_i)$ from the vectors $\pm(1, r_i)$ is contained in $\mathbf{U}_1^{(0)}$, $\xi \to 0$. Note that $v_1(\xi)$ has a unique critical number $k = 1$ satisfying the consistency condition [14] and $v_2(\xi)$ has no critical numbers.

The support of the expansion of the solution $v_1(\xi)$ agrees with $\mathbf{K} = \mathbb{N}$ and, hence,

$$v_1(\xi) = c_0 + \sum_{s=1}^{+\infty} c_s \xi^s,$$

where $c_s \in \mathbb{R}$; moreover, c_1 is chosen arbitrarily and the other coefficients c_{s+1} are uniquely determined through c_0 and c_1.

The support of the expansion of the solution $v_2(\xi)$ agrees with $\mathbf{K} = \{[(n+1)\sigma + m + 2n]/(\sigma+1),\ \mathbb{Z} \ni m, n \geq 0,\ m+n > 0\}$ and, hence,

$$v_2(\xi) = c_{\frac{\sigma}{\sigma+1}} \xi^{\frac{\sigma}{\sigma+1}} + \sum_{s \in \mathbf{K}} c_s \xi^s,$$

where the coefficients $c_s \in \mathbb{R}$ are uniquely determined through $c_{\sigma/(\sigma+1)}$.

REMARK 8. Since (4.6) has solutions with the expansions $v_2(\xi)$ as $\xi \to 0$, the initial problem with the conditions $v|_{\xi=0} = 0$ and $\lim_{\xi \to 0} v' = \pm\infty$ is solvable. This case conforms with (1.2); however, it is not considered in the framework of our article.

The vertices $\Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$ have no appropriate power solutions. In the first case the shortened equation does not have power solutions with exponents lying in the cone $\mathbf{U}_2^{(0)}$ and in the second case it is algebraic.

B. To the edge $\Gamma_1^{(1)}$, there correspond the shortened equation

$$\hat{f}_1^{(1)}(\xi, v) := vv'' + \frac{(v')^2}{\sigma} + \frac{v}{\omega} = 0 \tag{4.8}$$

and the normal cone $\mathbf{U}_1^{(1)} = \{\lambda(1,2),\ \mathbb{R} \ni \lambda > 0\}$. In this case $\xi \to \infty$, and (4.8) has a power solution of the form $v_3(\xi) = c_2\xi^2$, where $\mathbb{R} \ni c_2 \neq 0$. Inserting it into (4.8), we see that $c_2 = -\sigma/[2(\sigma+2)\omega]$.

Note that for $v_3(\xi)$ we have the two critical numbers $k_1 = -4/\sigma$ and $k_2 = 1$. The support of the expansion of the solution $v_3(\xi)$ agrees with $\mathbf{K} = \{2 - m,\ m \in \mathbb{N}\}$. Obviously, $k_2 \in \mathbf{K}$, and k_1 lies in $\mathbf{K}$ only if $\sigma = 4/l$, where $l \in \mathbb{N}$, and we have

$$v_3(\xi) = c_2\xi^2 + \sum_{s=-1}^{+\infty} c_{-s}\xi^{-s},$$

where $c_{-s} := c_{-s}(\log\xi)$ and c_1 and c_{k_1} contain an arbitrary constant while the remaining c_{-s} are determined uniquely.

If $\sigma \neq 4/l$ then $\mathbf{K}(k_1) = \{2 - m - 2(\sigma+2)n/\sigma,\ \mathbb{Z} \ni m, n \geq 0,\ m+n > 0\}$ and the expansion of the solution $v_3(\xi)$ is of the form

$$v_3(\xi) = c_2\xi^2 + \sum_{s \in \mathbf{K}(k_1)} c_{-s}\xi^{-s},$$

where $c_{-s} := c_{-s}(\log\xi)$; moreover, the coefficients c_1 and c_{k_1} contain an arbitrary constant while the remaining c_{-s} are defined uniquely.

C. To the edge $\Gamma_2^{(1)}$, there correspond the shortened equation

$$\hat{f}_2^{(1)}(\xi, v) := vv'' + \frac{(v')^2}{\sigma} + v' = 0 \tag{4.9}$$

and the normal cone $\mathbf{U}_2^{(1)} = \{-\lambda(1,1),\ \mathbb{R} \ni \lambda > 0\}$. In this case $\xi \to 0$ and (4.9) has a power solution of the form $v_4(\xi) = c_1\xi$, where $\mathbb{R} \ni c_1 \neq 0$. Inserting it into (4.9), we infer $c_1 = -\sigma$.

Note that $v_4(\xi)$ has no critical numbers. The support of the solution $v_4(\xi)$ agrees with $\mathbf{K} = \{m+1,\ m \in \mathbb{N}\}$ and, hence,

$$v_4(\xi) = c_1\xi + \sum_{s=2}^{+\infty} c_s\xi^s,$$

where the coefficients $c_s \in \mathbb{R}$ are determined uniquely through $c_1 = -\sigma$.

D. To the edge $\Gamma_3^{(1)}$, there correspond the shortened equation

$$\hat{f}_3^{(1)}(\xi, v) := v' + \frac{v}{\omega} = 0 \tag{4.10}$$

and the normal cone $\mathbf{U}_3^{(1)} = \{-\lambda(0,1),\ \mathbb{R} \ni \lambda > 0\}$. Equation (4.10) has no appropriate power solutions; however, it possesses a nonpower solution [14] of the form $v_5(\xi) = ce^{-\xi/\omega}$ as $\xi \to +\infty$, $\mathbb{R} \ni c \neq 0$.

The above analysis yields the following statement:

Lemma 4. *Solutions to* (4.1) *possess the following properties:*
1. *For* $v_1 = -\sigma$, *the asymptotic relations hold:*

$$v(\xi) = -\sigma\xi + o(\xi) \text{ as } \xi \to 0; \quad v(\xi) = -\frac{\sigma\xi^2}{2(\sigma+2)\omega} + o(\xi^2) \text{ as } \xi \to +\infty.$$

2. *For* $v_1 = 0$, *the asymptotic relations hold:*

$$\forall\, n \in \mathbb{N}\ v(\xi) = o(\xi^n) \text{ as } \xi \to -0; \quad v(\xi) = -\frac{\sigma\xi^2}{2(\sigma+2)\omega} + o(\xi^2) \text{ as } \xi \to -\infty.$$

4.4. The form of heat waves. Using the results obtained, we can study the behavior of solutions of the heat wave type with the logarithmic front in more detail.

For definiteness, we assume that a heat wave moves from the origin in the first octant; i.e., the function defining the motion of the front of a heat wave satisfies the relations $\mathscr{F}(t) = \log(c_1 t + c_2)^{\alpha\omega}$, $\mathscr{F}|_{t=0} = 0$, and $\mathscr{F}'|_{t=0} \geq 0$; hence, $c_2 = 1$ and $\alpha\omega c_1 > 0$. In this case (2.12) yields

$$u(t, \rho) = \frac{c_1 \alpha^2 \omega v(\xi)}{c_1 t + 1}, \tag{4.11}$$

where $\xi = \rho/\alpha - \omega \log(c_1 t + 1)$. Without loss of generality (see Section 4.1), we can assume that $\omega > 0$ and the study of the properties of a heat wave is reduced to the consideration of two possible cases.

I. Assume that $\omega > 0$, $\alpha > 0$, and $c_1 > 0$. Obviously, in this case $u(t, \rho) \geq 0$ if and only if $v(\xi) \geq 0$; thus we consider the nonnegative part of a solution to (4.1) for $v_1 = -\sigma$ (see Fig. 2(a)). From (4.11) and the arguments of Subsection 4.1, we have that the domain of the heat wave propagation at any time is defined by the inequalities $\log(c_1 t + 1)^{\alpha\omega} + \alpha\xi_0 \leq \rho \leq \log(c_1 t + 1)^{\alpha\omega}$, i.e., the wave has a rear front $\rho = \log(c_1 t + 1)^{\alpha\omega} + \alpha\xi_0$, where ξ_0 satisfies (4.4). In this case (4.11) yields $\lim_{t\to+\infty} u(t, \rho) = 0$ for all ρ.

II. Assume that $\omega > 0$, $\alpha < 0$, and $c_1 < 0$. In this case $0 \leq t < -1/c_1$ and $t = -1/c_1$ is a vertical asymptote of the front of a heat wave. Obviously, $u(t, \rho) \geq 0$ if and only if $v(\xi) \leq 0$ and the following particular cases are possible.

II.1. Assume that $v_1 = -\sigma$ and $v(\xi) \leq 0$ (see Fig. 2(a)). From (4.11) it follows that the domain occupied by a heat wave is defined by the inequality $\rho \leq \log(c_1 t + 1)^{\alpha\omega}$. The heating of the half-space till arbitrarily large temperature takes finite time and there is no localization of the heating impact. Similar configuration corresponds to the HS-regime with blow-up [10, p. 60] and the blow-up time is $0 < -1/c_1 < +\infty$.

II.2. Let $v_1 = 0$. Consider a nontrivial solution $v(\xi)$ to the Cauchy problem (4.1) with the zero initial data (see Fig. 2(b)). Since $\xi \leq 0$, (4.11) implies that the domain of the heat wave propagation is described as $\rho \geq \log(c_1 t + 1)^{\alpha\omega}$. This heat wave cannot be generated by any boundary regime for $\rho = 0$.

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