**International seminar**

# NUMERICAL SOLUTION OF INTEGRAL AND DIFFERENTIAL EQUATIONS

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**Lake Baikal**

**Abstracts**

**On one class of multistep methods for numerical solution of integral algebraic equations**

Budnikova O.S.
East Siberian State Academy of Education
Irkutsk, Russia
*e-mail:osbud@mail.ru*

Consider the system of integral equations

$$A(t)x(t) + \int_0^t g(t,s)K(t,s)x(s)ds = f(t),\ 0 \le s \le t \le 1, \qquad (1)$$

with the condition

$$det A(t) \equiv 0, \qquad (2)$$

where $A(t)$ and $K(t,s)$ are $(n \times n)$ matrices, while $f(t)$ and $x(t)$ are the given and the unknown $n$ -dimensional vector functions, respectively.

If $g(t,s) \equiv 1$, then such problems are called integral-algebraic equations (IAEs) with smooth kernels. If $g(t,s) = (t-s)^{-a},\ 0 < a < 1$, then the problem 1 with the condition 2 are called IAEs with weakly singular kernels.

In the report we give sufficient existence and uniqueness conditions for the problem 1 with the condition 2 for IAEs with smooth and weakly singular kernels.

For numerical solution of IAEs with smooth kernels we constructed multistep methods which are based on explicit Adam's methods for the integral term in 1 and extrapolation formulas for the first term:

$$A_{i+1}\sum_{j=0}^{k}\alpha_j x_{i-j} + h\sum_{l=0}^{i}\omega_{i+1,l}K_{i+1,l}x_l = f_{i+1}. \qquad (3)$$

And we modify methods 3 for numerical solution of IAEs with weakly singular kernels. We present results of numerical experiments.

This work was supported by the Russian Foundation for Basic Research, projects: 13-01-93002-V'et_a, 14-01-31224-mol_a.

**On non-classical difference schemes**

Bulatov M.V.
Institute for System Dynamics and Control Theory of SB RAS
Irkutsk, Russia
*e-mail: mvbul@icc.ru*

The approach is proposed for numerical solution of ODEs (initial problem). For original problem $L$ different $k$-step difference schemes are written out and combined into overdetermined system of finite-dimensional equations. In the general case the obtained system doesn't have solution. The method of reduction of this overdetermined system to system with the nonsingular square Jacobi matrix is proposed. This approach contains parameters those can be chosen to construct schemes having higher order and bigger stability region. The comparison of the method with well-known ones is carried out. The results of numerical calculations of model examples are given.

**The qualitative investigation of integral-algebraic equations**

Chistyakov V.F.
Institute for System Dynamics and Control Theory of SB RAS
Irkutsk, Russia
*e-mail: chist@icc.ru*

Consider systems of integral equations

$$A(t)x(t) + \int_{\alpha}^{t} p(t,s)K(t,s)x(s)ds = f,\ t \in T = [\alpha, \beta], \tag{1}$$

where $A(t),\ K(t,s)$ are$(m \times n)$ matrices, $x(t), f \equiv f(t)$ are unknown and given vector-function, respectively.

It is assumed that the input data are sufficiently smooth and the condition $p(t,s) = 1$ or $p(t,s) = 1/(t-s)^{\gamma},\ 0 < \gamma < 1$, is carried out.

In the report solvability conditions of systems (1) satisfying condition (2) are given.

$$\operatorname{rank} A(t) < \min\{m, n\},\ t \in T. \tag{2}$$

These results are continuation of investigation developed in [1].

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**Numerical solution of integral differential equation with singularities**

Thanh Do Tien
Irkutsk State Technical University
Irkutsk, Russia
*e-mail: thanhdotien278@yahoo.com*

The equation we need to solve is

$$t^{1-n}(t^{n-1}x'(t))' = f(x(t)), \tag{1}$$

where

$$f(x) = 4\lambda^2 x(x-\xi)(x+1), \tag{2}$$

$\xi, \lambda$ - some real parameters and $0 < \xi < 1$. The solution x determines an increasing mass density profile.

The boundary conditions for the bubbles:

$$x'(0) = 0, \tag{3}$$

$$\lim_{t\to\infty} x(t) = x_l, \tag{4}$$

where $x_l$ is the density of the external liquid.

Rewrite the problem (1) in the integral form:

$$x'(t) = \int_0^t \frac{\tau^{n-1} f(x(\tau))d\tau}{t^{n-1}}. \tag{5}$$

To solve the equation (5), quadrature formula of the first order and second order were used.

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**Implicit-explicit general linear methods for ordinary differential equations**

Zdzislaw Jackiewicz
School of Mathematical Statistical Sciences
Arizona State University
Tempe,Arizona,USA
*e-mail:jackiewicz@asu.edu*

For many systems of differential equations modeling problems in science and engineering, there are often natural splittings of the right hand side into two parts, one of which is non-stiff or mildly stiff, and the other part is stiff. For such systems we will develop a new class of implicit-explicit (IMEX) general linear methods (GLMs), where the stiff part is integrated by an implicit formula, and the non-stiff part is integrated by an explicit formula. We analyze convergence and stability of these methods when the implicit and explicit parts interact with each other. We will describe search for methods with large regions of absolute stability, assuming that the implicit part of the method is $A$- or $L$-stable. Finally we furnish examples of IMEX GLMs with optimal stability properties.

This is a joint work with A. Cardone from University of Salerno, and A. Sandu and H. Zhang from Virginia Polytechnic Institute State University.

**Strong stability preserving general linear methods**

Zdzislaw Jackiewicz
School of Mathematical Statistical Sciences
Arizona State University
Tempe,Arizona,USA
*e-mail:jackiewicz@asu.edu*

We describe the construction of strong stability preserving (SSP) general linear methods (GLMs) for ordinary differential equations. This construction is based on the monotonicity criterion for SSP methods. This criterion can be formulated as a minimization problem, where the ob- jective function depends on the Courant-Friedrichs-Levy (CFL) coefficient of the method, and the nonlinear constrains depend on the unknown remaining parameters of the methods. The solution to this constrained minimization problem leads to new SSP GLMs of high order and stage order.

This is a joint work with Giuseppe Izzo from the University of Naples.

**Analysis and numerical approximation of bubble-type solutions of singular nonlinear equations**

Pedro M. Lima
Centro de Matemaica e Aplicacoes , Instituto Superior Tecnico,
Universidade de Lisboa
Lisbon,Portugal
*e-mail:pedro.t.lima@ist.utl.pt*

We are concerned with a generalization of the Cahn-Hilliard continuum model for multiphase fluids [1], where the classical Laplacian has been replaced by a degenerate one (i.e., so-called p-Laplacian). Using spherical symmetry, the model can be reduced to a boundary value problem for a second order nonlinear ordinary differential equation. One searches for a monotone solution of this equation which satisfies certain boundary conditions. The case of the classical Laplacian was studied in detail in [2] and [3]. In the present work, it is proved that the arising boundary value problem, in the case of the p-Lapacian, possesses a unique strictly monotone solution. The asymptotic behavior of the solution is also analyzed at two singular points; namely, at the origin and at infinity. Efficient computational techniques for treating such singular boundary value problems are presented and the numerical results are discussed.

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**Non-classical method for stiff Volterra integral equations of the second kind**

Machkhina M.N.
East Siberian State Academy of Education
Irkutsk, Russia
*e-mail:masha888888@mail.ru*

Systems of second kind of Volterra integral equations with stiff and oscillating components are considered in this report. An implicit noniterative method of the second order is proposed for the numerical solution of such problems. The efficiency of the method is demonstrated using several examples.

This work was supported by the Russian Foundation for Basic Research, projects: 13-01-93002-V'et_a, 14-01-31224-mol_a.

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**Numerical solution of stiff ODE's by explicit methods**

Eugeny A. Novikov
Institute of computational modeling SB RAS
Krasnoyarsk, Russia
*e-mail:novikov@icm.krasn.ru*

In modeling kinetics of chemical reactions, calculation of electronic circuits, and other important applications, there is a problem of numerical solution of the Cauchy problem for stiff systems of ordinary differential equations. Main trends in construction of numerical methods are associated with expansion of their possibilities in solving problems of more and more high dimension. Mathematical formulation of practical problems become more and more accurate, which leads to increase of dimension and complexity of a right part of a system of differential equations. Despite improvement in computer performance, complexity of problems arising in practice outgrows development of computer technology, which in turn leads to increasing demands for computational algorithms.

Modern methods for solving stiff problems usually use calculation and inversion of the Jacobi matrix of a system of differential equations. In case of a sufficiently large dimension, efficiency of numerical methods is almost completely determined by inversion (decomposition) of the matrix. To increase efficiency of calculations in a number of algorithms, the freezing of the Jacobi matrix is used, that means using same matrix on several integration steps [1]. This approach is most successful in algorithms, based on multistep methods and, in particular, in backward differentiation formulas [2]. This problem does not cause any particular difficulties in constructing integration algorithms, based on other numerical schemes, if their stages are computed with the Jacobian matrix in some iterative process. This is due to the fact, that in this case the Jacobian matrix does not affect accuracy order of a numerical scheme, but only determines rate of convergence of iterations. So, it needs to be recomputed, when there is a significant slowdown in convergence rate of the iterative process.

The situation is worse in an integration algorithm, based on the known noniterative methods, which include methods of the Rosenbrock type [3] and their various modifications [1]. It should be noted that the noniteration method is much simpler in terms of computer implementation than algorithms based on numerical formulas, which are evaluated with using iterations. However, in methods of form [4], the Jacobi matrix affect accuracy order of a numerical scheme and, therefore, difficulties with its freezing arise. If a problem of using same matrix on several steps of integration is left unsolved, then, obviously one is limited to solve only problems of low dimensions. In [4,5], this problem is considered in relation to the Rosenbrock methods. It is proved, that maximum accuracy order of the Rosenbrock methods is equal to two, if in an integration algorithm the same Jacobi matrix is applied on several steps of integration. There is an algorithm with freezing Jacobi matrix, based on $L$-stable numerical formulas of second accuracy order and results of calculations, confirming its high efficiency.

Another important requirement for modern integration algorithms is numerical approximation of the Jacobi matrix. This is due to the fact that a right part of a system of differential equations often has large dimension and quite complex form. A typical example is provided by problems of chemical kinetics, where complexity of a right part side increases with number of elementary stages in a chemical reaction. Nowadays, simulation involves reactions, which contain dozens of reagents and hundreds of elementary stages. Therefore, in some cases, less effective numerical methods is more preferable, if their implementation does not require analytical calculation of elements of the Jacobi matrix. This barrier can be removed if an integration algorithm includes possibility of numerical approximation of the Jacobi matrix. Note, that the problem of freezing and numerical approximation are in some sense close to each other and, therefore, can be solved simultaneously.

Some analog of freezing the Jacobi matrix is using in calculations integration algorithms, based on explicit and $L$-stable methods with automatic selection of a numerical scheme. In this case, efficiency of the algorithm can be improved by calculating transitive regions corresponding to a maximum eigenvalue of the Jacobi matrix by an explicit method. It is natural to apply an inequality for stability control [6] as a criterion for choosing an efficient numerical formula. Note, that using such hybrid algorithms does not fully eliminate the problem of freezing the Jacobi matrix, because the explicit method can applied, generally speaking, only for a boundary layer solution, corresponding to a maximum eigenvalue of the Jacobi matrix.

On basis on explicit Runge Kutta type methods of high and low orders of accuracy principles of constructing algorithms of variable order and step are proposed for solution of moderately stiff problems. Algorithms with application of stages of Merson and Dormand-Prince methods are constructed. The results of calculations of examples [8] confirmed efficiency of algorithms of integration are shown.

This work was supported by the Russian Foundation for Basic Research (project 14-01-00047).

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**Computer-assisted proofs in floating-point**

Siegfried M. Rump
Institute for Reliable Computing
Hamburg University of Technology
Hamburg, Germany
*e-mail: rump@tu-harburg.de*

**Volterra integral equations of the first kind with piecewise continuous kernels**

Sidorov D.N.
Melentiev Energy Systems Institute of SB RAS
Irkutsk, Russia
*e-mail: contact.dns@gmail.com*

The system of linear Volterra integral equations of the first kind with piecewise continuous kernels is considered. The sufficient conditions (see [1,5]) for existence of unique solution of such system are obtained. In general case (solution can be non unique) we provide the method for construction of the asymptotic of solution. Under the conditions of existence and uniqueness theorem this problem is well-posed in sense of Hadamard on the pair of spaces $\left(\mathcal{C}_{[0,T]}, \quad \overset{\circ}{\mathcal{C}}{}^{(1)}_{[0,T]}\right)$ [5]. In scalar case (one equation only) the efficient numerical methods are suggested (see [2-5]). An efficiency of proposed methods are demonstrated on model examples.

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## On constructing test problem for stiff ODEs

Solovarova L.S.
Institute for System Dynamics and Control Theory of SB RAS
Irkutsk, Russia
*e-mail: soleilu@mail.ru*

The report deals with constructing autonomous linear system of ODE (initial problem), depending on parameter. Implicit Euler method is not effective for numerical solution of this example. The considerable restriction of integration step is required for convergence of this method. The restriction depends on input parameters. It turned out, that for proposed system some algorithms, for example two-stage Lobatto III C method, lose property of $L$-stability.

The constructing this example is based on the theory of numerical solution of differential algebraic equations of index 2.

**The characteristic vector exponent of the solution of the differential algebraic equations**

Ta Duy Phuong
Institute of Mathematics, VAST
Hanoi, Vietnam
*e-mail: tdphuong@math.ac.vn*

In the first part of talk, we consider the system

$$A(t)\dot{x} + B(t)x = f(t),\ t \in T_\infty = [0, \infty), \tag{1}$$

$$x(0) = a, \tag{2}$$

where $A$, $B$ are $(n \times n)$-matrix functions and $f$ is an $n$-vector function of $t$. For the system (1) we assume that

$$\det(A) \equiv 0. \tag{3}$$

Systems of the form (1) satisfying condition (3) are commonly called *differential algebraic equations* (DAEs). They play an important role in various applications. First we develope the concept of the vector characteristic exponents (introduced by Hoang Huu Duong for studying the stability of the ordinary differential equations) for DAEs (1). We also give some asymptotic stability criteria for DAEs (1).
In the second part, the concept of vector characteristic exponents are considered for the linear DAEs with properly stated leading terms of the following form

$$A(t)(D(t)x(t))' + B(t)x(t) = f(t),\ t \in T_\infty = [0, \infty). \tag{4}$$

This talk is based on joint work with Nguyen Thi Khuyen and Do Chung (see [1]-[3]).

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**Finite-Time Stability and Control of Linear Differential-Algebraic Equations with Time Delays**

Vu N. Phat
Institute of Mathematics, VAST
Hanoi, Vietnam
*e-mail: vnphat@math.ac.vn*

In this report, the problem of finite-time stability and stabilization for linear differential-algebraic equations (LADEs) with tiem delay is addressed. Fisrt, without resorting to the decomposition and equivalent transformation of the singular equation and using new estimation technique developed for linear singular system, some new delay-dependent criteria are established for the considered systems to be regular, impulse, and robustly finite-time stable. Then, based on Lyapunov-Krasovskii function method and linear matrix inequality (LMI) technique, new sufficient conditions formulated in terms of LMI which ensure the finite-time stability of the closed-loop LDAEs are derived. Numerical examples are given to illustrate the efficiency of the proposed methods.